Analysis of $L_{2N,2}$

$$F^0 = \begin{array}{c|c} & \begin{matrix} \phi_1 & \phi_2 & \cdots & \phi_N & \phi_{N+1} & \cdots & \phi_{2N} \end{matrix} \\ \begin{matrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \\ \phi_{N+1} \\ \phi_{N+2} \\ \vdots \\ \phi_{2N} \end{matrix} & \left[\begin{array}{ccccc|c} 1 & 0 & \cdots & & 0 & \\ 1 & 0 & \cdots & & 0 & \\ 0 & 1 & \cdots & & 0 & 0 \\ & 0 & 1 & \ddots & & \\ & & & 1 & 0 & \\ \hline & & & & & 1 \;\; 0 \;\; \cdots \;\; 0 \\ & & & & & 1 \;\; 0 \;\; \cdots \;\; 0 \\ & & 0 & & & 0 \;\; 1 \;\; \cdots \;\; 0 \\ & & & & & 0 \;\; 1 \;\; \ddots \\ & & & & & 1 \;\; 0 \end{array}\right] \end{array}$$

$$F^1 = \left[\begin{array}{ccccc|c} 0 & 1 & \cdots & & 0 & \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & 1 & \ddots & \\ 0 & 0 & \cdots & & 1 & \\ 0 & 0 & \cdots & & 0 & 1 \\ \hline & & & & & 0 \;\; 1 \;\; \cdots \;\; 0 \\ & & 0 & & & 0 \;\; 0 \;\; 1 \;\; \cdots \;\; 0 \\ & & & & & 0 \;\; 0 \;\; \cdots \;\; 1 \\ & & & & 1 & 0 \;\; 0 \;\; \cdots \;\; 0 \end{array}\right]$$

Assume $d_i = 1 - c_i \quad i = 1, 2.$

$$\tilde{F} = \left[\begin{array}{cccccc|cccccc} d_1 & c_1 & 0 & \cdots & & 0 & 0 & \cdots & & & & 0 \\ d_1 & 0 & c_1 & \cdots & & 0 & 0 & \cdots & & & & 0 \\ 0 & d_1 & 0 & c_1 & \cdots & 0 & 0 & \cdots & & & & 0 \\ \vdots & & & & & c_1 & & & & & & \\ 0 & & \cdots & & d_1 & 0 & & \cdots & & & & c_1 \\ \hline 0 & 0 & \cdots & & & 0 & d_2 & c_2 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & & & 0 & d_2 & 0 & c_2 & \cdots & & 0 \\ 0 & 0 & \cdots & & & 0 & 0 & d_2 & 0 & c_2 & \cdots & 0 \\ \vdots & & & & & & & & & & & \\ 0 & & \cdots & & & c_2 & 0 & & \cdots & & d_2 & 0 \end{array}\right]$$

$d_1 = 1 - c_1$

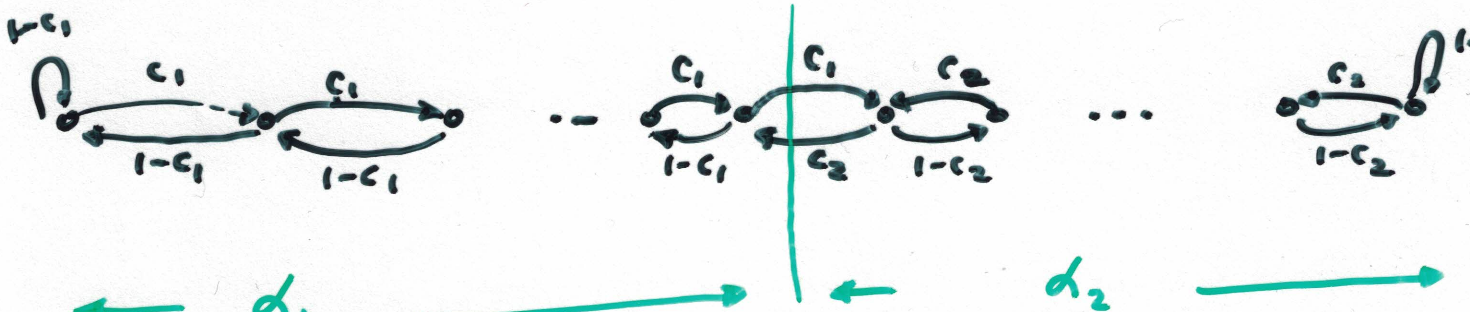
$d_2 = 1 - c_2$

WHAT is OVERALL M.C.

$$\underline{\pi}^* = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{2N} \end{bmatrix} \quad \underline{\text{satisfies}} \quad \underline{\pi}^* = \tilde{F}^T \pi^*$$

subject to $\sum_{i=1}^{2N} \pi_i = 1.$

SINCE ERGODIC M.C.



$d_1 \pi_1 + d_1 \pi_2 = \pi_1$ ← Bdy Cond → $d_2 \pi_{N+1} + d_2 \pi_{N+2} = \pi_{N+1}$

$c_1 \pi_1 + d_1 \pi_3 = \pi_2$ $\qquad$ $c_2 \pi_{N+1} + d_2 \pi_{N+3} = \pi_{N+2}$

$\vdots$

$c_1 \pi_{K-1} + d_1 \pi_{K+1} = \pi_K$ ← Genl. Equation $\qquad$ $c_2 \pi_{N+K-1} + d_2 \pi_{N+K+1} = \pi_{N+K}$ ← Genl. Eq

$\vdots$

$c_1 \pi_{N-1} + c_2 \pi_{2N} = \pi_N$ ← Bdy Cond → $c_2 \pi_{2N-1} + c_1 \pi_N = \pi_{2N}$

$\Downarrow$

General Solution.

chterstic Eqn.

$d_1 \lambda^2 - \lambda + c_1 = 0$

Roots. $1, \ \frac{c_1}{d_1} = e_1$

Solution

$\pi_K = A_1 e_1^{K-1} + B_1$

$e_1 = c_1/d_1$

$\Downarrow$

Genl. Solution

$d_2 \lambda^2 - \lambda + c_2 = 0$

Roots $1, \ \frac{c_2}{d_2} = e_2.$

$\pi_{N+K} = A_2 e_2^{K-1} + B_2$

$e_2 = \frac{c_2}{d_2}$

$d_1 \pi_{k+1} - \pi_k + c_1 \pi_{k-1} = 0$

<u>Charectistic Eqn</u>

$d_1 \lambda^2 - \lambda + c_1 = 0$ (Two roots)

<u>2 roots</u>. $\lambda_1 = 1. \Rightarrow d_1 . 1 - 1 + c_1 = 0 = c_1 + d_1 = 1.$

$\lambda_2 = \frac{c_1}{d_1} \Rightarrow d_1 \left(\frac{c_1}{d_1}\right)^2 - \left(\frac{c_1}{d_1}\right) + c_1$

$= \cancel{d_1} \frac{c_1^2}{d_1^{\cancel{2}}} - \frac{c_1}{d_1} + c_1$

$= c_1^2 - c_1 + c_1 d_1 = 0$

$= -c_1 \underbrace{(1 - c_1)} + c_1 \underbrace{(1 - \cancel{d_1})} = 0$

$\therefore \frac{c_1}{d_1}$ is a root.

<u>General Sol^n.</u>

$\pi_k = A_1 \left(\frac{c_1}{d_1}\right)^{k-1} + B_1 (1)^{k-1}$

$\frac{c_1}{d_1} = e_1$

$$\boxed{\pi_k = A_1 . e_1^{k-1} + B_1}$$

$e_2 = \frac{c_2}{d_2}$

$$\boxed{\pi_{N+k} = A_2 . e_2^{k-1} + B_2}$$

$$\pi_k = A_1 e_1^{k-1} + B_1 \qquad k = 1, \dots N$$

i.e. $\pi_1 = A_1 + B_1$

$\pi_2 = A_1 e_1 + B_1$

Bdy Cond. $d_1 \pi_1 + d_1 \pi_2 = \pi_1$

i.e. $d_1 \left(A_1 (1 + e_1) + 2B_1\right) = A_1 + B_1$

But $1 + e_1 = \frac{1}{d_1}$ $\qquad 1 + \frac{c_1}{d_1}$

$\therefore \quad A_1 + 2d_1 B_1 = A_1 + B_1 \Rightarrow \underline{\underline{B_1 = 0}}$

$$\boxed{\begin{array}{l} \therefore \ \pi_k = A_1 e_1^{k-1} \\ \text{lll}^{r}\text{ly} \quad \pi_{N+k} = A_2 e_2^{k-1} \end{array}} \qquad 0 \le k \le N$$

$$\pi_K = A_1 e_1^{K-1} \qquad \text{for ACTION 1}$$

$$\pi_{N+K} = A_2 e_2^{K-1}$$

$$\therefore \; p_1 = \Pr(\text{choosing } d_1) = \sum_{K=1}^{N} \pi_K = A_1 \sum_{K=1}^{N} e_1^{K-1}$$

$$= \frac{A_1 (e_1^N - 1)}{e_1 - 1}$$

$$p_2 = \Pr(\text{choosing } d_2) = \frac{A_2 (e_2^N - 1)}{e_2 - 1}.$$

Since $p_1 + p_2 = 1$.

$$\frac{A_1 (e_1^N - 1)}{e_1 - 1} + \frac{A_2 (e_2^N - 1)}{e_2 - 1} = 1 \qquad (*)$$

To get value of A_1 in terms of A_2.

USE OTHER BDY COND.

$$c_2 \pi_{2N-1} + c_1 \pi_N = \pi_{2N}$$

$$c_2 A_2 e_2^{N-2} + c_1 A_1 e_1^{N-1} = A_2 e_2^{N-1}$$

group terms, ..., $A_1 e_1^N d_1 = A_2 e_2^N d_2$

$$\frac{A_1}{A_2} = \frac{e_2^N d_2}{e_1^N d_1}. \qquad (**$$

SUBSTITUTE (**) IN (*)

WE SOLVE FOR A_1.

THEN TRIVIALLY A_2 IS OBTAINED ...

<u>THUS</u> p_1 AND p_2 CAN BE OBTAINED.

$$p_1(\infty) = \frac{1}{1 + \left(\frac{c_1}{c_2}\right)^N \frac{c_1 - d_1}{c_2 - d_2} \frac{(c_2^N - d_2^N)}{(c_1^N - d_1^N)}}$$

$$p_2(\infty) = \frac{1}{1 + \left(\frac{c_2}{c_1}\right)^N \frac{c_2 - d_2}{c_1 - d_1} \frac{(c_1^N - d_1^N)}{(c_2^N - d_2^N)}}$$

<u>THUS</u> $M(\infty)$: $c_1\, p_1(\infty) + c_2\, p_2(\infty)$ is

$$M(\infty) = \frac{\frac{1}{c_1^{N-1}} \frac{c_1^N - d_1^N}{c_1 - d_1} + \frac{1}{c_2^{N-1}} \frac{c_2^N - d_2^N}{c_2 - d_2}}{\frac{1}{c_1^N} \frac{c_1^N - d_1^N}{c_1 - d_1} + \frac{1}{c_2^N} \frac{c_2^N - d_2^N}{c_2 - d_2}}.$$