Consider $R=2$. (2 action case)

\[ p_{i}(n+1) = (1-a) \cdot p_{i} \]

\[ = 1 - p_{2} + b \cdot p_{2} \]

\[ = 1 - (1-a) \cdot p_{2} \]

\[ = (1-b) \cdot p_{i} \]

if \( d = d_{2} \), \( B = 0 \)

if \( d = d_{2} \), \( B = 1 \)

if \( d = d_{1} \), \( B = 0 \)

if \( d = d_{1} \), \( B = 1 \)

General $L_{R,p}$ (Linear Reward-Penalty Scheme)

\[ d = d_{2}, B = 1 \]

\[ p_{2} \leftarrow (1-b) \cdot p_{2} \]

\[ p_{1}(n+1) = 1 - (1-b) \cdot p_{2} \]

\[ = 1 - p_{2} + b \cdot p_{2} \]

\[ \therefore p_{1} \leftarrow 1 - p_{2} = 1 - (1-b) \cdot p_{2} \]

\[ d = d_{1}, B = 0 \]

\[ p_{2} \leftarrow (1-b) \cdot p_{2} \]

\[ p_{1} \leftarrow 1 - p_{2} = 1 - (1-a) \cdot p_{2} \]
Simple Case. \( a = b. \) (Symmetric \( L_a, p \) Scheme).

\[
\begin{align*}
    p_{i+1} &= kp_i & \text{if } d = d_1, \beta = 1 \\
    &= 1 - kp_i & \text{if } d = d_1, \beta = 0 \\
    &= 1 - kp_i & \text{if } d = d_2, \beta = 0 \\
    &= kp_i & \text{if } d = d_2, \beta = 0
\end{align*}
\]

Note: \((d_2, \beta = 0) \equiv (d_1, \beta = 1)\)

And: \((d_1, \beta = 0) \equiv (d_2, \beta = 1)\)

Note: \(p_i, p_{i+1}\) are both random variables.

Previously: \(P(n) = F^TP(n-1)\) (PSSA)

Now we can't do that.

\(p_i(n+1)\) can take 4 values - depending on outcome.

i.e. \(p_i(n+1)\) is a r.v. with a mean, variance etc.
What is \( E[p_{i(n+1)}] \)? Use only \( p_i \) in expression.

\[

\begin{align*}
p_{i(n+1)} &= kp_i & \text{w.p. } p_i c_i \\
&= 1 - k (1-p_i) & \text{w.p. } p_i (1-c_i) \\
&= 1 - k (1-p_i) & \text{w.p. } (1-p_i) c_2 \\
&= kp_i & \text{w.p. } (1-p_i) (1-c_2)
\end{align*}
\]

\[
E[p_{i(n+1)}] \text{ is a function of } p_i
\]

\[
E[p_{i(n+1)} | p_i] = kp_i \left( p_i c_1 + (1-p_i) (1-c_2) \right)
\]

\[
+ (1 - k (1-p_i)) \left( p_i (1-c_1) + (1-p_i) c_2 \right)
\]

Note: All \( p_i^2 \) terms luckily cancel.

\[
E[p_{i(n+1)} | p_i] = p_i - k (c_1+c_2) p_i + k c_2
\]

\[
E[p_{i(n+1)}] = \left[ 1 - k (c_1+c_2) \right] E[p_i] + k c_2.
\]

A linear difference equation...

\[
y_{i(n+1)} = ey_i + f.
\]
General Solution.

In the limit, \( p_i(\infty) = p_i(n) \)

\[
\bar{p}_i(n+1) = (1 - k(c_i + c_j)) \bar{p}_i(n) + k c_2
\]

\[
\bar{p}_i(\infty) = (1 - k(c_i + c_j)) \bar{p}_i(\infty) + k c_2
\]

\[ k(c_i + c_j) \bar{p}_i(\infty) = R c_2 \]

\[ \therefore E[p_i(\infty)] = \frac{c_2}{c_i + c_j} \]

Theorem.

1. The symmetric \( LR, p \) scheme has

\[ E[p_i(\infty)] = \frac{c_j}{c_i + c_j} \]

if \( i \neq j \), \( i, j = 1, 2 \)

and is therefore expedient.

2. The limiting exp. value of \( p_i(\infty) \) is independent of the parameter of \( LR, p \) scheme.
Is there a "better" way to look at this?  

Yes!

$$E[p_i(n+1) \mid p_i] = p_i - K (C_i + C_2) p_i + K C_2.$$  

I.e. writing it in terms of both $p_i$ & $p_2$.

$$(K C_2 = (p_1 + p_2) (K C_2))$$

$$E[p_i(n+1) \mid p_i] = p_i [1 - K (C_i + C_2) + K C_2] + K C_2 p_2$$

$$= p_i (1 - K C_1) + K C_2 p_2.$$  

$$E[p_i(n+1)] = (1 - K C_1) E[p_i] + K C_2 E[p_2]$$

$$E[p_2(n+1)] = K C_1 E[p_i] + (1 - K C_2) E[p_2]$$

$$E[p_i(n+1)] = \begin{bmatrix} 1 - K C_1 & K C_2 \end{bmatrix} \begin{bmatrix} E[p_i] \\ E[p_2] \end{bmatrix}$$

I.e. $E[p_i]$ are the "states of an ergodic M.C."
\[ F = \begin{bmatrix}
1 - kC_1 & kC_1 \\
RC_2 & 1 - kC_2
\end{bmatrix} \]

**Markov matrix**

**Solution** is
\[ \begin{bmatrix}
\frac{C_2}{C_1 + C_2} \\
\frac{C_1}{C_1 + C_2}
\end{bmatrix} \]

Just as we thought...

i.e. \[ E[p_1(n)] \rightarrow \frac{C_1}{C_1 + C_2} \]

**Eigenvalues**:
- Always one is unity.
- Other: \[ (1 - kC_1 + 1 - kC_2) = 1 + \lambda_2 \]
  \[ \lambda_2 = 1 - k(C_1 + C_2) \]

Rate of convergence determined by 'k'

**Variance**
\[ \text{Var} p_1(n) = \frac{C_1 C_2}{(C_1 + C_2)^2} \cdot \frac{(1 - k)^2}{(1 - k)^2 + 2k(1 - k)(C_1 + C_2)} \]