Angle Orders, Regular n-gon Orders
and the
Crossing Number of a Partial Order

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Abstract: A finite poset $P(X,<)$ on a set $X=\{x_1,...,x_m\}$ is an angle order (regular n-gon order) if the elements of $P(X,<)$ can be mapped into a family of angular regions on the plane (a family of regular polygons with n sides and having parallel sides) such that $x_i<x_j$ if and only if the angular region (regular n-gon) for x_i is contained in the region (regular n-gon) for x_j. In this paper we prove that there are partial orders of dimension 6 with 64 elements which are not angle orders. The smallest partial order previously known not to be an angle order has 198 elements and has dimension 7. We also prove that partial orders of dimension 3 are representable using equliateral triangles with the same orientation. This result does not generalizes to higher dimensions. We will prove that there is a partial order of dimension 4 with 14 elements which is not a regular n-gon order regardless of the value of n. Finally, we prove that partial orders of dimension 3 are regular n-gon orders for n≥3.

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1. Introduction

Let □$=\{S_1,...,S_m\}$ be a family of sets. A partial order $P(X,<)$ on a set $X=\{x_1,...,x_m\}$ represents □ if $x_i<x_j$ in $P(X,<)$ iff S_i is contained in S_j, $i\neq j$. □ is called a set representation of $P(X,<)$. Partial orders arising from specific families of sets have been studied in various papers in the literature. For instance partial orders of dimension 2 have set representations using intervals of the real line. (See [3]). Partial orders of dimension 2 can also be represented using families of circles on the plane; to see this, one can take a representation □$=\{I_1,...,I_m\}$ of a poset $P(X,<)$ of dimension 2 using intervals along the x-axis, then for each interval I_j of □ build a circle C_j such that I_j is a diameter of C_j . It is easily seen that □$'=\{C_1,...,C_m\}$ is a set representation for $P(X,<)$. When the elements of □ are boxes in the n-dimensional space $\mathbb{R}^n$ (i.e. sets of points $\{\mathbf{x}=(x_1,...,x_n) : a_i\leq x_i\leq b_i;\ a_i,\ b_i$ constants, $i=1,...,n\}$) the posets thus obtained are exactly all 2n-dimensional posets. (See [5]). Partial orders of dimension 2n also have set representations using convex n-polygons on the plane. (See [10]). When the elements of □ are arcs on a circle, we obtain circular permutation graphs. (See [8]).

In this paper we will study partial orders arising from families of circles (circle orders), angular regions in the plane (angle orders) and regular polygons with n sides (regular n-gon orders). Angle orders were first studied in Fishburn and Trotter [4]. They showed that all posets of dimension 4 and all interval orders are angle orders. They presented a partial order of dimension 7 and 198 points which is not an angle order. In section 2 of this paper we present a partial order of dimension 6 with 64 elements which is not an angle order. This, however, does not solve their problem of deciding whether there are dimension 5 posets which are not angle orders.

In section 3 we study regular n-gon orders, that is posets arising from families of regular n-gons *all of which have the same orientation* , that is, all of which have parallel sides. We prove that all posets of dimension 3 are representable by families of equilateral triangles. This result is tight in the sense that for dimension $n>3$, we prove that there are posets of dimension n which are not regular n-gon orders. More surprisingly, we will prove that there are posets of dimension 4 which are not regular n-gon orders *regardless of the value* of n. Finally we prove that posets of dimension 3 are regular n-gon orders for every value of n. This provides good evidence towards the validity of the following conjecture: Every partial order of dimension 3 is a circle order. Several results in this paper are proved by using the crossing number of partial orders. The crossing number of partial orders defined in [6] has been useful in the study of geometric comtainment problems. (See [10]).

1.2 Terminology and Definitions

A binary relation < over a set X defines a *partial order* P(X,<) on X if it satisfies

(i) $x<y$, $y<z$ implies $x<z$ (transitivity), and

(ii) $x<x$ (antisymmetry).

The partially ordered set P(X,<) is a *linear order* if it also satisfies

(iii) $x<y$ or $y<x$ for all distinct $x,y \in X$.

Let P(X,<) be a poset. A *realizer* of P of size $k+1$ is a collection of linear orders $\{L_o(X,<_0), L_1(X,<_1),..., L_k(X,<_k)\}$ such that $L_o(X,<_0) \cap L_1(X,<_1) \cap ... \cap L_k(X,<_k)=P(X,<)$. where the intersection is defined by

$$x<y \text{ iff } x<_i y \text{ for all } i.$$

It can be easily proved that every poset can be obtained as the intersection of a number of linear orders. Dushnik and Miller [2] define the *dimension* of P, denoted dim P, to be the size of the smallest possible realizer of P. Such a realizer is called a *minimum realizer* of P.

1.3 Function Diagrams

Let $\mathcal{F}=\{f_1,\dots,f_m\}$ be a family of continuous functions $f_i:[0,1]\to \mathbb{R}$, $i=1\dots m$. The family $\mathcal{F}=\{f_1,\dots,f_m\}$ is called *normal* if the following conditions are satisfied:

a) For any pair of elements $f_i, f_j \in \mathcal{F}$, $i\neq j$, the set of values $S(i,j)=\{x \in [0,1] : f_i(x)=f_j(x)\}$ is finite.

b) $f_i(0)\neq f_j(0)$, $f_i(1)\neq f_j(1)$; $i\neq j$.

c) Each time the graphs of two different functions intersect, they cross each other; that is if $f_i(x_0)=f_j(x_0)$ there exists an $\varepsilon>0$ such that $x_0-\varepsilon<x< x_0<y< x_0+\varepsilon$ implies that $f_i(x)<f_j(x)$ and $f_i(y)>f_j(y)$ or $f_i(x)>f_j(x)$ and $f_i(y)<f_j(y)$.

Informally speaking, a set of functions $\mathcal{F}=\{f_1,\dots,f_m\}$ is *normal* if the graphs of any two elements $f_i, f_j \in \mathcal{F}$ intersect a finite number of times and each time they intersect, they cross each other.

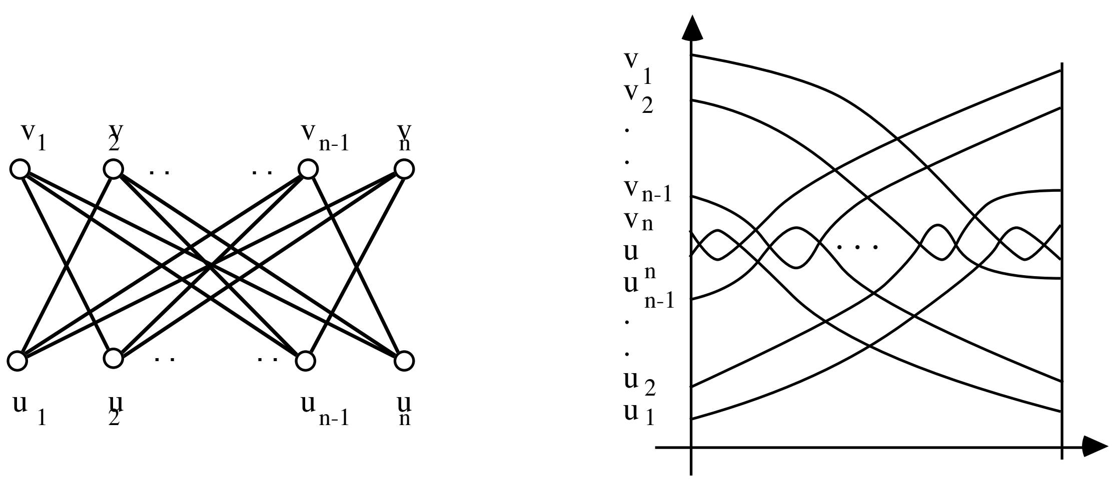
Let $X=\{x_1,\dots,x_m\}$ be a set, and $P(X,\prec)$ a partial order on X. $P(X,\prec)$ is called a function order (f-order for short) if there exists a normal set of functions $\mathcal{F}=\{f_1,\dots,f_m\}$ such that $x_i \prec x_j$ if $f_i(x)<f_j(x)$ for all $x \in [0,1]$. The set of functions $\mathcal{F}=\{f_1,\dots,f_m\}$ will be called an f-diagram for $P(X,\prec)$. We will also say that $P(X,\prec)$ represents $\mathcal{F}$. It is easy to prove that every poset is an f-order. (See [6]).

1.4 The Crossing Number of a Partial Order

Given an f-diagram □=$\{f_1,\ldots,f_m\}$, the *crossing number* □(□) is defined as the maximum over the set $\{|S(i,j)|: f_i, f_j$ □ □, $i\neq j\}$; that is the maximum number of times two elements of □ intersect. The *crossing number* □(P(X,<)) *of a poset* P(X,<) is now defined as min{□(□): □ is an f-diagram for P(X,<)}.

Informally speaking, every partial order can be represented in many ways using a normal set □=$\{f_1,\ldots,f_m\}$ of continuous real functions with domain [0,1]. In each such representation, the graphs of some elements of □ intersect a number of times. The crossing number of a poset P(X,<) is k if in any f-diagram □=$\{f_1,\ldots,f_m\}$ representing P(X,<) there are at least two elements of □=$\{f_1,\ldots,f_m\}$ that intersect at least k times. Notice that if □(P(X,<))=0, then P(X,<) has an f-diagram □ in which no pair of functions of □ intersect, thus P(X,<) is a linear order. It is also easy to prove that if □(P(X,<))=1, then the dim P(X,<) is 2 and that in general □(P(X,<))≤ dim P(X,<) -1. (See [6]).

Let □$_n$(X,<) be the poset with elements $X=\{u_1,\ldots,u_n, v_1,\ldots,v_n\}$ such that $u_i<v_j$, $i\neq j$, and all other pairs of elements in □$_n$(X,<) are not comparable. □$_n$(X,<) is called the Hiraguchi poset. It is well known that the dimension of □$_n$(X,<) is n. In [6] it was proved that the crossing number of the Hiraguchi poset □$_n$(X,<) is 2, $n\geq 3$. (See figure 1).



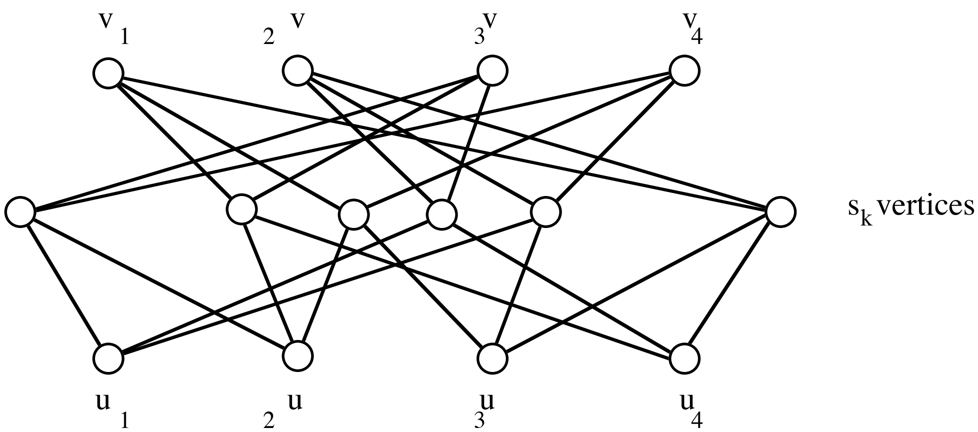
(a) (b)

Figure 1.

Let □$_n$ be the poset obtained from □$_n$(X,<) as follows: For each subset S_k of $\{1,\ldots,n\}$ with exactly □n/2□and □(n+1)/2□elements (if n is even both values are the same, if n is odd they are

different), insert in $\square_n(X,<)$ a new element s_k such that $s_k>u_j$, $j \in S_k$, $s_k< v_i$, $i \notin S_k$; $s_i< s_j$ if $S_i< S_j$, $i\neq j$.
(See figure 2). The next results were proved in [10]:

Theorem 1: The crossing number $\square(\square_n)= n-1$, dim $\square_n=n$.



$\square_4$

Figure 2.

2.1 Representations of Partial Orders Using Regular Convex Polygons

Let $\mathcal{S}=\{S_1,\ldots,S_n\}$ be a family of sets. A partial order $P(X,<)$ on a set $X=\{x_1,\ldots,x_n\}$ represents $\mathcal{S}=\{S_1,\ldots,S_n\}$ if S_i is contained in S_j implies $x_i<x_j$ in $P(X,<)$. Conversely $\mathcal{S}=\{S_1,\ldots,S_n\}$ will be called a set representation of $P(X,<)$. In this section we study the problem of representing partial orders using convex polygons on the plane. If no restrictions are imposed on the polygons to be used we can easily prove the following result:

Theorem 2. Every poset has a representation using convex polygons on the plane.

Proof: Let $P(X,<)$ be a poset on a set $X=\{x_1,\ldots,x_n\}$. Let S be a convex polygon with n vertices. Label the vertices of S using the elements $x_1,\ldots,x_n$ of X in the clockwise direction along the boundary of S. For every i let $S_i=\mathrm{Conv}(\{x_j \in X : x_j< x_i\} \cup \{x_i\})$, i.e. the convex closure of $S_i=\{x_j \in X : x_j< x_i\} \cup \{x_i\}$. It follows easily that $\{S_1,\ldots,S_n\}$ is a set representation of $P(X,<)$.

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In fact, it can be provedthat all 2n-dimensional posets are n-gon orders, that is, for every partial order P(X,<) of dimension 2n, there is a set representation using convex polygons with n sides [10]. For example, partial orders of dimension 6 are triangle orders. From now on, we shall assume that *a point cannot be a vertex of more than one polygon* of a polygon representation of a partial order and that any two edges of different polygons *intersect at most in one point*.. In this section we will study partial orders arising from families of regular polygons *with parallel sides* ; for instance we will consider families of equilateral triangles with bases parallel to the x-axis.The next result was our original motivation to study these families of partial orders.

Theoem 3. Every poset of dimension 3 can be represented using equilateral triangles.

Proof: Let P(X,<) be a poset of dimension 3 on a set X={x_1, .., x_n} and L_1, L_2, L_3 linear extensions of P(X,<) such that L_1 □ L_2 □ L_3=P(X,<). Take 3 rays R_1, R_2, R_3 emanating from the origin at 120° angles. Label n points in R_i using the elements of X in the order determined by L_i, i=1,2,3. For a point x_j in R_i let $L_{i,j}$ be the line perpendicular to R_i through x_j, i=1,2,3, j=1,...,n. Each such line determines a semiplane $S_{i,j}$ containing the origin. For each x_i□X let T_i=$S_{1,j}$ □ $S_{2,j}$ □ $S_{3,j}$. It follows that T_i is contained in T_i if and only in T_i< T_i in P(X,<). (See figure 3).

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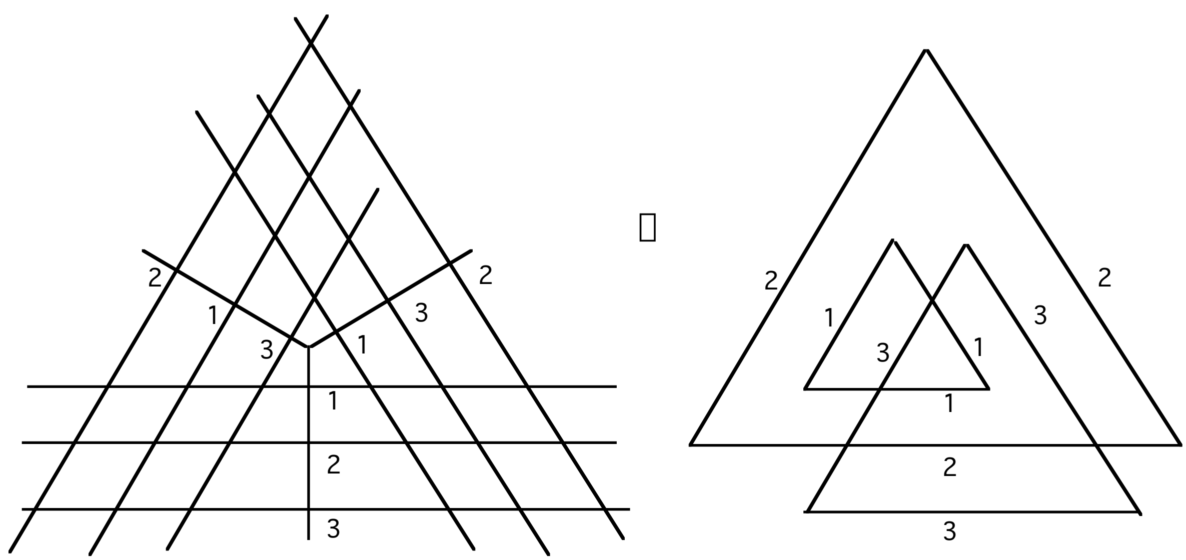

Figure 3.

Thus a natural question to ask is what posets can be represented using regular n-gons. For instance: What posets can be represented using equilateral triangles, squares, pentagons...? Can we extend the result in Theorem 3 to higher dimensions? Unfortunately the answer to the last question is negative. In fact we will prove that □$_4$ is not representable using regular n-gons regardless of the value of n. We recall that in this section, we shall be concerned only with regular n-gons *all having the same orientation*. A representation $\mathcal{R}=\{S_1,\ldots,S_n\}$ of a partial order P(X,<) is called *normal* if $S_1 \cap \ldots \cap S_n \neq \emptyset$.

A poset will be called a *regular n-gon order* if it can be represented using regular n-gons all with the same orientation. Our first result concerning regular n-gon orders is this:

Lemma 1: Any regular n-gon order has a normal representation.

Proof: Let $\mathcal{R}=\{P_1, .., P_k\}$ be a representation of an regular n-gon order P(X,<) on a set $X=\{x_1, .., x_k\}$ using regular n-gons. Suppose that all elements of $\mathcal{R}$ are contained in a circle of radius 1. For each element P_i of $\mathcal{R}$ let P_i' be the regular n-gon obtained from P_i as follows: For each edge e_j of P_i let S_j be the semiplane containing P_i defined by the line L_j parallel to e_j at distance 1 from e_j, j=1,...,n. Let $P_i'= S_1 \cap ... \cap S_n$. It follows immediately that $R'=\{P_1',... P_k'\}$ is a normal representation of P(X,<). (See figure 4).

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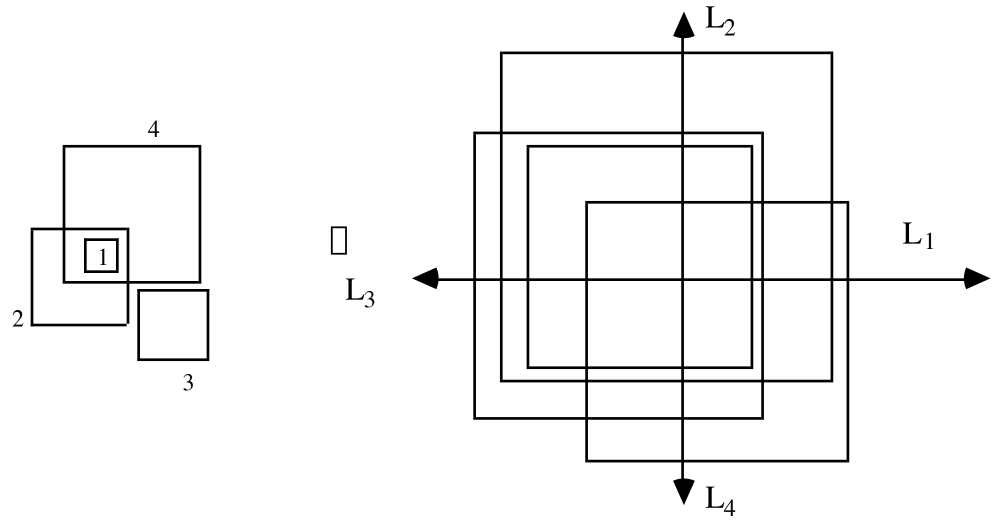
We can now prove

Theorem 4: The dimension of every regular n-gon order is at most n. Moreover, there are regular n-gon orders with dimension n.

Proof: Let $\mathcal{R}=\{P_1, .., P_k\}$ be a normal representation of a regular n-gon order P(X,<). Suppose that the origin belongs to the common intersection of $P_1,...,P_k$. For each edge e_i of P_1 let R_i be the ray emanating from the origin that intersects e_i perpendicularly. Each R_i defines a linear extension L_i of P(X,<) (the order in which the elements of $\mathcal{R}$ are intersected by R_i), i=1,...,n. It is easy to see that $L_1,...,L_n$ is a realizer of P(X,<). (See figure 4).



$L_1=\{1,2,4,3)\}$, $L_2=\{3,1,2,4\}$
$L_3=\{3,1,4,2\}$, $L_4=\{1,4,2,3\}$

Figure 4.

Next we shall prove that the Hiraguchi poset H_n is a regular n-gon order set. We will prove this result for the case n=3. The proof can easily be adapted for other values of n. Take an equilateral triangle T on the plane. Let us extend each edge e_i of T at both ends by a constant to obtain e'_i, i=1,2,3. Using e'_i let us construct an equilateral triangle T_i containing T, one of whose edges is e'_i. It is easy to see that T_i is well defined, i=1,2,3. In the perpendicular to the mid-point of e_i, we can always choose a point p_i not contained in T_i, i=1,2,3. Then the partial order representing $\{T_1, T_2, T_3, p_1, p_2, p_3\}$ is H_3. (See figure 5).

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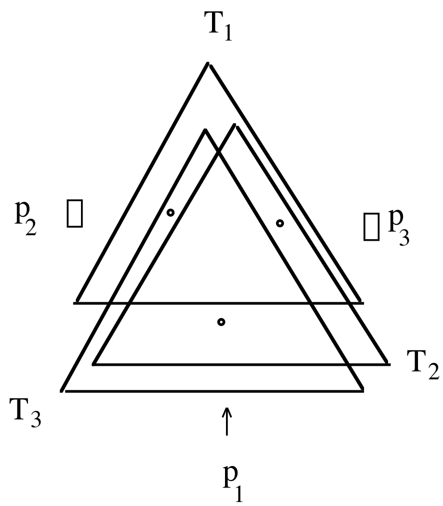

Figure 5.

We now proceed to prove that there are partial orders of dimension 4 which are not regular n-gon orders regardless of the value of n. In order to prove this, we need the next lemma.

Lemma 2: Let P(X,<) be a regular n-gon order ($n \geq 3$) . Then the crossing number of P(X,<) is at most two.

Proof: Let $\mathcal{F}=\{P_1,...,P_n\}$ be a normal representation of P(X,<) using regular n-gons all with the same orientation. Let p be a point in $P_1 \cap ... \cap P_n$. Let L_p be a ray emanating from p that does not meet any point in which the boundaries of any two elements of $\mathcal{F}$ intersect. Then using what in topology is known as surgery, we can cut the plane along L_p and stretch it so that one side of the cut goes to the y-axis and the other to the line x=1. In doing so, the boundary of each P_j is mapped into a continuous function $f_i[0,1] \to \mathbb{R}$. Then we obtain an f-diagram $\mathcal{F}=\{f_1,\ldots,f_m\}$ for P(X,<) with crossing number at most 4. (See figure 6).

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Theorem 5. There are posets with dimension n>3 which are not regular n-gon orders.

Proof: By theorem 1 there are posets of dimension n and crossing number n-1, $n \geq 1$. For n>3 these posets have crossing number $n-1 \geq 3$, and then by Lemma 2 can not be represented using regular n-gons.

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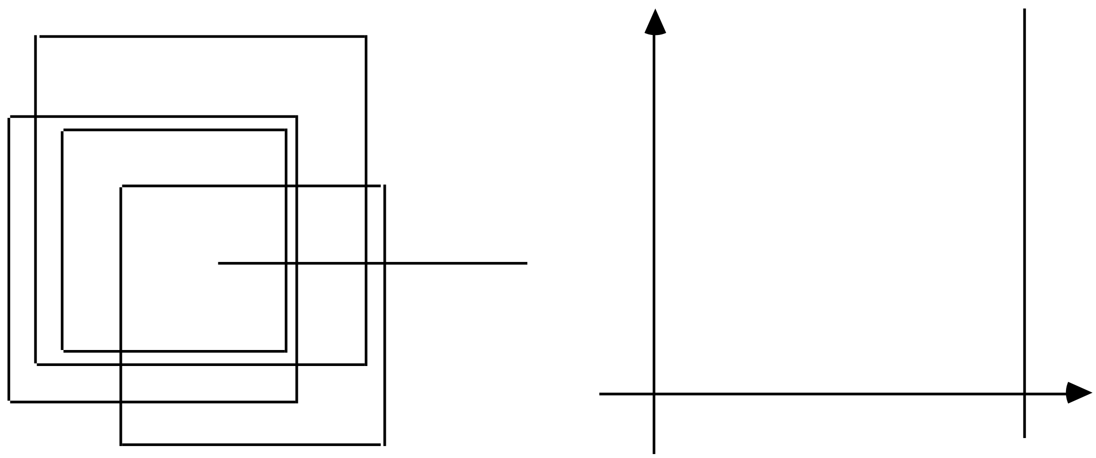


Figure 6.

Corollary 1: $\square_4$ is not a regular n-gon order regardless of the value of n.

$\square_4$ is the smallest partial order known to us that is not a regular n-gon order. It has only 14 elements. We strongly believe that $\square_4$ is the smallest ordered set that is not a regular n-gon order.

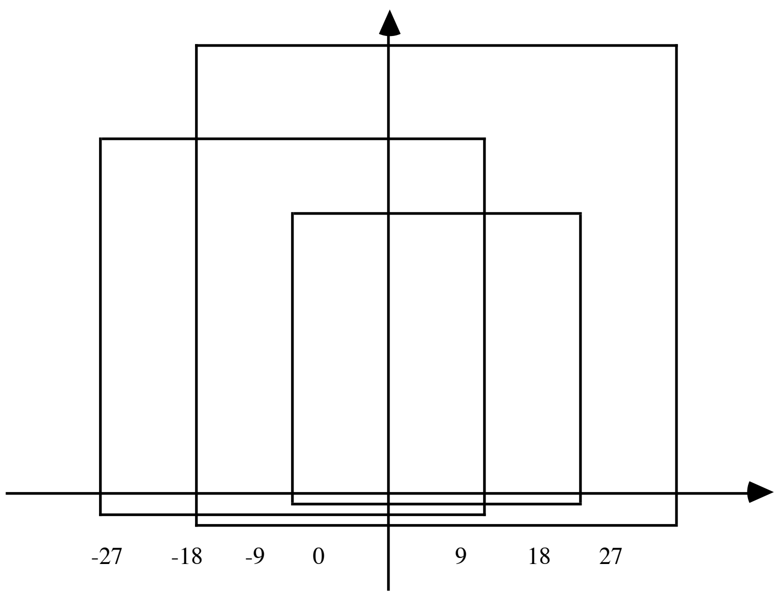
2.2 Dimension 3

Let $X=\{x_1,\ldots,x_m\}$ be a set, and $P(X,<)$ a partial order on X. $P(X, <)$ is called a circle order if there exists a family $\square=\{P_1,\ldots,P_m\}$ of circles on the plane such that $x_i < x_j$ iff circle P_i is contained inside P_j. One of the most interesting areas in the study of geometric containment problems is the study of circle orders. In [10] it was proved that the crossing number of circle orders is at most 2. Since $\square_4$ has crossing number 3, $\square_4$ is not a circle order. On the other hand, all partial orders with dimension 2 are circle orders. Then from a dimension point of view, the standing problem is that of deciding if all partial orders with dimension 3 are circle orders. This seems to be a very hard problem. In this section, we shall present a result which provides good evidence that the answer to the above problem is positive. We will prove that partial orders with dimension 3 are regular n-gon orders, regardless of the value of n, $n\geq 3$. Notice that as n□ the polygons thus obtained converge to circles. Unfortunately, we cannot apply a limit type argument to our result to solve the circle orders problem.

Theorem 6. Every partial order with dimension 3 is a regular n-gon order, $n\geq 3$.

Proof: As in previous cases, we will prove our result for n=4. For larger values of n, the proof can be easily adapted to obtain the desired result. Let P(X,<) be a partial order of dimension 3 on X={1,...,m} .

Let $\{L_1(X,<_1), L_2(X,<_2), L_3(X,<_3)\}$ be a realizer of P(X,<). Suppose w.l.o.g. that $L_1(X,<_1)$ is the linear extension in which i $<_1$ j iff as integers i<j. Each $L_i(X,<_i)$ defines a permutation π_i on {1,...,m}, i=2,3. Using π_2, label the points with coordinates (k m^2,0) on the x-axis with the element $\pi_2^{-1}(k)$, k=1,...,m. Similarly, label the points (-k m^2,0) on the x-axis with the element -$\pi_3^{-1}(k)$, k=1,...,m. (See figure 7).



L_1={1,2,3} , L_2={1,3,2} , L_3={2,1,3}

Figure 7.

For each i let □(i) be the line segment contained in the line y=i determined by the perpendiculars to the x-axis at the two points labelled i. Each □(i) uniquely determines a square S(i) with base □(i), i=1,...,m. Notice that i<j in P(X,<) imply :

a) The base □(j) of S(j) is at least $2m^2$ units longer than □(i).

b) The projection of □(j) on the x-axis contains the projection of □(i).

c) □(j) is at most m-1 units below □(i).

From a), b) and c) it follows immediately that S(i) is contained in S(j).It is now easy to verify that □$=\{S(1),...,S(m)\}$ is a representation for $P(X,<)$.

For the general case, instead of using the perpendiculars to the x-axis at the two points labelled i, we use a line forming a $2\pi/n$ angle with the positive x-axis at the point labelled i and another line at a $(n-2)\pi/n$ angle with the negative x-axis at the point labelled -i. The points on the y-axis should also be repositioned closer to the x-axis.

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3.1. Angle orders

We now proceed to study representations of partial orders using families of angular regions. Angle orders were introduced in Fishburn and Trotter [4]. An *angular region* is a closed region A of $\mathbb{R}^2$ bounded by a pair of rays R_1 and R_2 emanating from a point p containing all points swept out by rays from p in the clockwise direction from R_1 to R_2. A poset $P(X,<)$ is called an angle order if it has a representation using angular regions in the plane. In [4] it was proved that all posets of dimension at most four are angle orders. Moreover, in the same paper it was proved that the poset consisting of all elements $x \in 2^7$ (under containment) with at most four elements is not an angle order. Such poset has 198 elements. The following questions was posed in [4]:

Problem 1: What is the dimension of the least dimension poset which is not an angle order?

Problem 2: Is 2^5 an angle order? And in general what is the smallest n such that 2^n is not an angle order?

In the rest of this section, we shall produce a poset of dimension 6 with 64 elements which is not an angle order. Thus problem 1 reduces to the following: Are all posets of dimension 5 angle orders? The problem of deciding if 2^5 is an angle order remains open. The technique used to prove our results follows some ideas presented in [4] combined with the crossing number of posets.

Let A be an angular region bounded by two rays R_1 and R_2. If the angle between R_1 and R_2 is less than 180°, A is called a *little* angle; if the angle exceeds 180°, A will be called a *big* angle. The point p will be called the *vertex* of A. Notice that if the angle between R_1 and R_2 is 180° then the vertex of A is not unique.

An angle order $P(X,<)$ that has a representation using only little angles will be called an *l-angle order*. If $P(X,<)$ has a representation using only big angles, $P(X,<)$ will be called a *b-*

angle order.

For a given angular region A in the plane let $A^*=\mathbb{R}^2-A$. Clearly if A is a little angular region, A* is a big angular region. The next result follows immediately from observations made in [4].

Lemma 3 [4]: If P(X,<) is an l-angle (b-angle) order, then the dual P*(X,<) of P(X,<) is a b-angle (l-angle) order.

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Next, we observe that the boundaries of two different angular regions intersect in at most four points. This lead us to the following result:

Lemma 4: The crossing number of an l-angle order is at most 4.

Proof: Let R= $\{A_1,...,A_n\}$ be a representation of P(X,<) using little angular regions. Assume without loss of generality that all the intersection points of the boundaries of the angular regions of R are contained in a circle of radius 1 with center in the origin.

Let us assume that $x_1< x_2<...<x_n$ is a linear extension of P(X,<). For each element $x_i \in X$ let S_i be the region determined by the intersection of A_i with the circle of radius i and center in the origin. Clearly $x_i<x_j$ if and only if S_i is contained in S_j, $i\neq j \in \{1,...,n\}$. Let $S_i(1)$ be the set of points in $\mathbb{R}^2$ at distance ≤1 from S_i. Since the origin belongs to $S_i(1)$,i=1,...,n, $S_1 \cap ... \cap S_n \neq Ø$. Moreover, the boundaries of any two such sets intersect in at most four points. Let L_0 be a ray emanating from the origin that does not meet any point in which the boundaries of any two elements $S_i(1)$, $S_j(1)$ intersect. Using surgery again as in Lemma 2 our result follows.

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Lemma 5: The crossing number of a b-angle order is at most 4.

Proof: By lemma 3 the dual P*(X,<) of an angle order P(X,<) is an l-angle order. Thus the crossing number of P*(X,<) is at most 4. But the crossing number of a poset is equal to the crossing number of its dual. The result now follows.

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The next result trivially follows:

Lemma 6: Let P(X,<) be a poset with crossing number greater than or equal to 5. Then if P(X,<) is an angle order, any representation $R=\{A_i,..., A_i\}$ of P(X,<) contains a little and a big angular region.

Theorem 6: There are posets of dimension 6 which are not angle orders.

Proof: The idea used in the proof of this theorem is similar to the one used by Fishburn and Trotter in their proof of Corollary 2 in [4]. Let P(X,<) be any poset with crossing number at least 5. Let us construct a new poset Q consisting of two isomorphic copies P_1(X,<), P_2(X,<) of P(X,<) such that if x $\in$ P_1(X,<) and y $\in$ P_2(X,<) then x<y. We claim that Q is not an angle order. For if Q is an angle order, then there is a representation R of Q using angular regions in the plane. By lemma 6 there is a big angular region A_i representing an element $x_i \in P_1$(X,<). Similarly there is a small region B_j representing an element $y_j \in P_2$(X,<). However, since by definition $x_i < y_j$, A_i is contained in B_j which is impossible. Letting P(X,<)= $\square_6$ gives us the desired result.

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Since $\square_6$ has 12+C(6,3) elements, Theorem 6 gives us a poset with 64 elements which is not an angle order. Some open problems are now presented. We proved that the crossing number of l-angle orders and b-angle orders is at most 4. This leads us to the following question:

Is it true that the crossing number of angle orders is at most 4? More specifically, is $\square_6$ an angle order? So far we have been unable to verify if $\square_6$ is an angle order or not.

What about partial orders with crossing number ≤4; is it true that any partial order with crossing number at most 4 is an angle order?

Similar questions can be asked about regular n-gon orders. For instance, is it true that all partial orders with crossing number 2 are regular n-gon orders for some n?

Remark: It has been called to our attention that W. T. Trotter has been able to use arguments similar to those presented here to obtain a five-dimensional order which is not a circle order. If 2^5 is a little angle order, then its crossing number is at most 3, a contradiction. The result now follows by using 2^5 in Theorem 6 instead of $\square_6$.

Alon and Scheinermann have also been able to prove that there are partial orders of dimension 5 which are not angle orders [1]

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