Distributed Computing by Mobile Robots: Solving the Uniform Circle Formation Problem

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Abstract. Consider a set of $n \neq 4$ simple autonomous mobile robots (decentralized, asynchronous, no common coordinate system, no identities, no central coordination, no direct communication, no memory of the past, deterministic) initially in distinct locations, moving freely in the plane and able to sense the positions of the other robots. We study the primitive task of the robots arranging themselves equally spaced along a circle not fixed in advance (Uniform Circle Formation). In the literature, the existing algorithmic contributions are limited to restricted sets of initial configurations of the robots and to more powerful robots. The question of whether such simple robots could deterministically form a uniform circle has remained open. In this paper, we constructively prove that indeed the Uniform Circle Formation problem is solvable for any initial configuration of the robots without any additional assumption. In addition to closing a long-standing problem, the result of this paper also implies that, for pattern formation, asynchrony is not a computational handicap, and that additional powers such as chirality and rigidity are computationally irrelevant.

1 Introduction

Consider a set of punctiform computational entities, called *robots*, located in $\mathbb{R}^2$ where they can freely move. Each entity is provided with a local coordinate system and operates in *Look-Compute-Move* cycles. During a cycle, a robot obtains a snapshot of the positions of the other robots, expressed in its own coordinate system (*Look*); using the snapshot as an input, it executes an algorithm (the same for all robots) to determine a destination (*Compute*); and it moves towards the computed destination (*Move*). After a cycle, a robot may be inactive for some time.

To understand the nature of the distributed universe of these mobile robots and to discover its computational boundaries, the research efforts have focused on the minimal capabilities the robots need to have to be able to solve a problem. Thus, the extensive literature on distributed computing by mobile robots has almost exclusively focused on very simple entities operating in strong adversarial conditions. The robots we consider are *anonymous* (without ids or distinguishable features), *autonomous* (without central or external control), *oblivious* (no recollection of computations and observations done in previous cycles), *disoriented* (no agreement among the individual coordinate systems, nor on unit distance and chirality). In particular, the choice of individual coordinate systems, the activation schedule, the duration of each operation during a cycle, and the length traveled by a robot during its movement are determined by an adversary; the only constraints on the adversary are fairness (i.e., for every time t and each robot r there exists $t' > t$ when r is active), finiteness (i.e., the duration of each activity and inactivity is arbitrary but finite), and minimality (i.e., there exists $\delta > 0$, unknown to the robots, such that if the destination is at distance at most δ the robot will reach it, else it will move at least δ towards the destination, and then it may be unpredictably stopped by the adversary). For this type of robots, depending on the activation schedule and timing assumptions, three main models have been studied in the literature: the *asynchronous* one ($\mathcal{A}$SYNC), where no assumptions are made on synchronization among the robots' cycles nor their duration, and the *semi-synchronous fully synchronous* models, denoted by $\mathcal{S}$SYNC and $\mathcal{F}$SYNC, respectively, where the robots, oblivious and disoriented, however operate in synchronous rounds, and each round is "atomic": all robots active in that round terminate their cycle by the next round; the only difference is whether all robots are activated in every round ($\mathcal{F}$SYNC), or, subject to some fairness condition, a possibly different subset is activated in each round ($\mathcal{S}$SYNC). All three models have been intensively studied (e.g., see [1–3, 5–10, 15–17, 23, 24]; for a detailed overview refer to the recent monograph [13]).

The research on the *computability* aspects has focused almost exclusively on the fundamental class of Geometric Pattern Formation problems. A *geometric pattern* (or simply *pattern*) P is a set of points in the plane; the robots *form* the pattern P at time t if the configuration of the robots (i.e., the set of their positions) at time t is similar to P (i.e., coincident with P up to scaling, rotation, translation, and reflection). A pattern P is *formable* if there exists an algorithm that allows the robots to form P within finite time and no longer move, regardless of the activation scheduling and delays (which, recall, are decided by the adversary) and of the initial placement of the robots in distinct points. Given a model, the research questions are: to determine if a given pattern P is formable in that model; if so, to design an algorithm that will allow its formation; and, more in general, to fully characterize the set of patterns formable in that model. The research effort has focused on answering these questions for $\mathcal{A}$SYNC and the less restrictive models both in general (e.g., [5, 15, 16, 22–24]) and for specific classes of patterns (e.g., [1, 7, 8, 10–12, 19, 20]).

Among specific patterns, a special research place is occupied by two classes: `Point` and `Uniform Circle`. The class `Point` is the set consisting of a single point; point formation corresponds to the important Gathering problem requiring all robots to gather at a same location, not determined in advance (e.g., see [2–4, 18, 21]). The other important class of patterns is `Uniform Circle`: the points of the pattern form the vertices of a regular n-gon, where n is the number of robots (e.g., [1, 6–8, 10–12, 20]).

In addition to their relevance as individual problems, the classes `Point` and `Uniform Circle` play another important role. A crucial observation, by Suzuki and Yamashita [23], is that formability of a pattern P from an initial configuration Γ in model $\mathcal{M}$ depends on the relationship between $\rho_{\mathcal{M}}(P)$ and $\rho_{\mathcal{M}}(\Gamma)$, where $\rho_{\mathcal{M}}(V)$ is a special parameter, called *symmetricity*, of a multiset of points V, interpreted as robots modeled by $\mathcal{M}$. Based on this observation, it follows that the only patterns that *might* be formable from any initial configuration in $\mathcal{F}$SYNC (and thus also in $\mathcal{S}$SYNC and $\mathcal{A}$SYNC) are single points and uniform circles. It is rather easy to see that both points and uniform circles can be formed in $\mathcal{F}$SYNC, i.e. if the robots are fully synchronous. After a long quest by several researchers, it has been shown that GATHERING is solvable (and thus `Point` is formable) in $\mathcal{A}$SYNC (and thus also in $\mathcal{S}$SYNC) [2], leaving open only the question of whether `Uniform Circle` is formable in these models. In $\mathcal{S}$SYNC, it was known that the robots can *converge* towards a uniform circle without ever forming it [7]. Some recent results indicate that the robots can actually form a uniform circle in $\mathcal{S}$SYNC. In fact, by concatenating the algorithm of [19], for forming a biangular configuration, with the one of [11], for circle formation from an equiangular starting configuration, it is possible to form a uniform circle starting from any initial configuration in $\mathcal{S}$SYNC; notice that the two algorithms can be concatenated only if the robots are semi-synchronous. Hence, the only outstanding question is whether it is possible to form a uniform circle in $\mathcal{A}$SYNC. In spite of the simplicity of its formulation and the repeated efforts by several researchers, the existing algorithmic contributions are limited to restricted sets of initial configurations of the robots and to more powerful robots. In particular, it has been proven that, with the additional property of *chirality* (i.e., a common notion of "clockwise"), the robots can form a uniform circle [12], and with a very simple algorithm; the fact that `Uniform Circle` is formable in $\mathcal{A}$SYNC $+$*chirality* follows also from the recent general result of [16]. The difficulty of the problem stems from the fact that the inherent difficulties of asynchrony, obliviousness, and disorientation are amplified by their simultaneous presence.

In this paper we show that indeed the UNIFORM CIRCLE FORMATION problem is solvable for any initial configuration of $n \neq 4$ robots without any additional assumption, closing a problem open for over a decade. This result also implies that, for GEOMETRIC PATTERN FORMATION problems, *asynchrony* is not a computational handicap, and that additional powers such as *chirality* and *rigidity*[5] are computationally irrelevant.

2 Definitions

For a finite set $S \subset \mathbb{R}^2$ of $n > 2$ points, we define the *Smallest Enclosing Circle*, or *SEC*, to be the circle of smallest radius such that every point of S lies on the circle or in its interior. For any S, SEC is easily proven to exist and to be unique. Three other circles will play a special role: these are concentric with SEC, and

[5] A move is *rigid* if it is not interrupted before reaching the destination point.

have radiuses that are 1/2, 1/3, and 1/4 the radius of SEC. They are denoted by *SEC/2*, *SEC/3*, and *SEC/4*, respectively.

The *angular distance*, with respect to point x, between two points p and q (distinct from x) is the measure of the smallest angle between $\angle pxq$ and $\angle qxp$, and is denoted by $\theta_x(p,q)$. The *sector* defined by two points a and b is the locus of points c such that $\theta_x(a,c)+\theta_x(c,b)=\theta_x(a,b)$. Whenever x is not specified, it is assumed to be the center of the SEC of a well-understood set of points.

Given a finite set S, the positions of its points around some point $x \notin S$, taken clockwise, naturally induce a cyclic order on S. If several points of S lie on the same ray emanating from x, their relative order is induced by their distance from x, starting from the nearest point.

Let $p_0 \in S$ be any point, and let $p_i \in S$ be the $(i+1)$-th point in the cyclic order around $x \notin S$, starting from p_0. Let $\alpha_x^{(i)} = \theta_x(p_i, p_{i+1})$, where the indices are taken modulo n. Then, $(\alpha_x^{(i)})_{0 \leqslant i < n}$ is called the *angle sequence* induced by p_0. (Of course, depending on the choice of $p_0 \in S$, there may be at most n different angle sequences with respect to x.) Letting $\beta_x^{(i)} = \alpha_x^{(n-i)}$, for $0 \leqslant i < n$, we call $(\beta_x^{(i)})_{0 \leqslant i < n}$ the *reverse angle sequence* induced by p_0. We let $\widetilde{\alpha}_x$ and $\widetilde{\beta}_x$ be, respectively, the lexicographically smallest angle sequence and the lexicographically smallest reverse angle sequence of S. Also, we denote by μ_x the lexicographically smallest between $\widetilde{\alpha}_x$ and $\widetilde{\beta}_x$, and by $\mu_x^{(i)}$ the i-th element of μ_x. If $p \in S$ is any point inducing μ_x as a clockwise or counterclockwise angle sequence, we say that p is a *lex-first* point of S (with respect to x), and we denote by $\mathcal{L}_1$ the set of all lex-first points. Let p be a lex-first point of S and suppose that μ_x is the clockwise (resp. counterclockwise) angle sequence induced by p. Let p' be the first point after p in the clockwise (resp. counterclockwise) order around x that is not collinear with x and p. Then, p' is said to be a *lex-second* point of S (with respect to x), and we denote by $\mathcal{L}_2$ the set of all lex-second points. If x is not specified, it is assumed to be the center of the SEC of S.

The following definitions apply whenever the symbols used are well defined, i.e., if and only if no point of S lies in the center of SEC. S is *co-radial* if $\mu^{(0)} = 0$. In a co-radial set, every two points at angular distance 0 are said to be *co-radial* with each other. The number of distinct clockwise angle sequences of S (with respect to the center of its SEC) is called the *period* of S. It is easy to verify that the period is always a divisor of n.

We will be distinguishing among different types of configurations, defined below (see also Figure 3). S is said to be *Equiangular* if its period is 1, *Biangular* if its period is 2, and *Aperiodic* if its period is n. In a *Biangular* set, any two points at angular distance $\mu^{(0)}$ are called *neighbors*, and any two points at angular distance $\mu^{(1)}$ are called *quasi-neighbors*. If a *Biangular* configuration is not co-radial, it is called *Simple biangular*. An *Aperiodic* configuration can be *Uni-aperiodic* if $\widetilde{\alpha} \neq \widetilde{\beta}$, and *Bi-periodic* if $\widetilde{\alpha} = \widetilde{\beta}$. A set S that is not *Aperiodic* is said to be *Uni-periodic* if $\widetilde{\alpha} \neq \widetilde{\beta}$, and *Bi-periodic* if $\widetilde{\alpha} = \widetilde{\beta}$. S is *Regular* if its points are the vertices of a regular n-gon.

We say that point $p \in S$ is *homologous* to point $q \in S$ if the angle sequence induced by p is equal to the angle sequence induced by q, or to its reverse. In

Algorithm UNIFORM CIRCLE FORMATION

Find first match of observed configuration:

1. **Regular:** Do nothing;
2. **Central:** Execute CENTRAL;
3. **Equiangular:** Execute EQUIANGULAR;
4. **Pre-regular:** Execute PRE-REGULAR;
5. **Pre-equiangular:** Execute PRE-EQUIANGULAR;
6. **Landmark-co-radial:** Execute LANDMARK-CO-RADIAL;
7. **Post-periodic:** Execute POST-PERIODIC;
8. **Antipodal-referees:** Execute ANTIPODAL-REFEREES;
9. **Simple Biangular:** Execute SIMPLE BIANGULAR;
10. **Periodic:** Execute PERIODIC;
11. **Post-aperiodic:** Execute POST-APERIODIC;
12. **Aperiodic:** Execute APERIODIC;

Fig. 1. The UNIFORM CIRCLE FORMATION algorithm

particular, if it is equal to the angle sequence induced by q (and not necessarily to its reverse), p and q are said to be *analogous*. Homology and analogy are equivalence relations on S, and the equivalence classes that they induce on S are called *homology classes* and *analogy classes*, respectively. In a *Uni-periodic* set of period k, all homology classes are *Equiangular* sets of size n/k. In a *Bi-periodic* set of period k, each homology class is either a *Biangular* set of size $2n/k$, or an *Equiangular* set of size n/k or $2n/k$. In a *Uni-aperiodic* set, the homology classes consist of one point; in a *Bi-aperiodic*, they consist of either one or two points.

S is said to be *Double-biangular* if it is *Bi-periodic* with period 4 and has exactly two homology classes.

S is *Pre-regular* if there exists a regular n-gon (called the *supporting polygon*) such that, for each pair of adjacent edges, one edge contains exactly two points of S (possibly on its endpoints), and the other edge's relative interior contains no point of S [8]. There is a natural correspondence between points of S and vertices of the supporting polygon: the *matching vertex* v of point $p \in S$ is such that v belongs to the edge containing p, and the segment vp contains no other point of S. If two points of S lie on the same edge of the supporting polygon, then they are said to be *companions*.

Finally, S is *Central* if one of its points lies at the center of SEC.

3 The Algorithm

3.1 High-Level Description

The general idea of the algorithm is that first some robots identify themselves as *referees* (in spite of anonymity) and maintain their role until they are the only ones not in their final position. The referees univocally determine special points, the *landmarks*, which, in turn, define a set of half-lines from the centre of SEC, the *targets*, partitioning the plane in n equal sectors. Each robot is assigned a different target. By positioning themselves on the targets, the robots reach an *Equiangular* configuration, and they ultimately form a uniform circle.

Algorithm UNIFORM CIRCLE FORMATION (see Figure 1) consists of an ordered set of tests to determine the class of the current configuration; this determines which action is going to be taken by a robot in order to implement the general strategy described above. The universe of possible configurations is decomposed by the algorithm into several classes. Some of the classes (i.e., *Regular*, *Central*, *Equiangular*, *Pre-regular*, *Simple biangular*, *Periodic*, *Aperiodic*) have been defined in Section 2; the others will be defined in the following, along with the description of the corresponding actions. It is easy to see that all possible configurations are covered by these classes, simply because any configuration is either *Periodic* or *Aperiodic*. Hence, if all other tests fail, one of these two necessarily succeeds.

We stress that some configurations belong to more than one class, and so the order in which such classes are tested by the algorithm matters. For instance, a *Pre-regular* configuration may easily be also *Aperiodic*. The reason why *Pre-regular* is tested before *Aperiodic* is that, when the robots execute procedure PRE-REGULAR and the configuration remains *Pre-regular* but it also accidentally becomes *Aperiodic*, we want all robots to keep executing the same procedure, without letting some of them "erroneously" start executing procedure APERIODIC. Of course, now the opposite problem may arise: when the robots are executing procedure APERIODIC, they may accidentally form a *Pre-regular* configuration. However, as it will be apparent in later sections, this event is much less likely, and it is easier to predict and handle by the algorithm in such a way that, if a *Pre-regular* configuration may be formed accidentally during the execution, then all robots agree to stop in that configuration and consequently start executing procedure PRE-REGULAR in a synchronized fashion.

3.2 Basic Tools

The above high-level description gives an idea of the general *intended behavior* of the robots. Asynchrony and special configurations can easily make the algorithm deviate from this behavior. The rules and movements of the robots are carefully designed so to handle any deviation, and they are quite complex. In particular, two tools are employed: cautious moves and special circles.

Cautious Move. If a robot's movement can potentially create some configuration that would be treated by other observing robots in an inconsistent way (i.e., a configuration of a class tested *before* the current one by the algorithm), the rule will prescribe the robot to stop in the first point that might create it. We call these points *critical points*. Thus in some procedures of the algorithm, robots are specifically required to perform an operation called *cautious move*; this method is invoked when there is a set of robots that need to move on disjoint paths, each of which contains finitely many critical points. It is assumed that, as the robots move along their paths, the set of critical points does not change.

In a cautious move, first the set of critical points is expanded with a set of "auxiliary" critical points: if a robot has a critical point on its path, located at distance d from the endpoint of the path (where the distance is measured along the path itself), then each other robot whose path is not shorter than d acquires a new critical point at distance d from the end of its path. The last point along each robot's path is also taken as a critical point.

Then, each robot r whose remaining path is longest moves forward along its path by the greatest possible amount, with the following constraints:

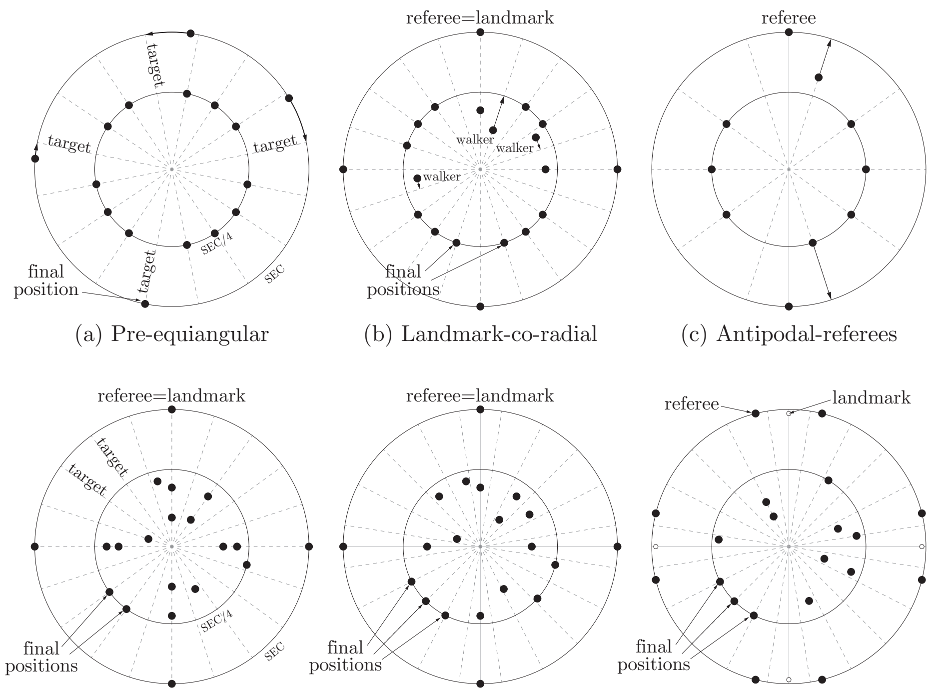
– r's destination point must not be past the next critical point (auxiliary or not);
– if r is currently lying on a critical point (auxiliary or not), its destination point must be at most halfway toward the next critical point (auxiliary or not) along its path;
– if the remaining path of r has length d, and there is another robot whose remaining path has length $d' < d$, then r's destination point must be at most d' away from the endpoint of r's path (in other words, robots do not "pass each other" in one turn).

On the other hand, the robots whose remaining path is not longest wait.

Special Circles. In the algorithm we use specific concentric circles: SEC, SEC/2, SEC/3, and SEC/4. This is done first of all to facilitate the recognition of the current configuration and coordinate the operations of the robots. For example, SEC/4 is used in *Periodic* while SEC/3 is used in *Aperiodic*. More importantly, these circles are used to *avoid* the accidental formation of certain configurations. In particular, as long as some robots are on or inside SEC/3, a *Pre-regular* configuration may never be formed: this is crucial in the proof of correctness of the algorithm. Note that we assume the robots can perform "circular movements" when the destination point is along one of these circles, but, at the cost of slightly modifying the algorithm, it is possible to let the robots move only along straight lines.

3.3 The Initial Tests

The first four tests performed by the algorithm are the simplest ones. The algorithm first checks if a uniform circle has been formed; if so, no further action is taken. Otherwise, it checks if there is a robot at the centre of SEC. In this case, that robot moves, avoiding collisions, to become co-radial with the robots on one of the most populated radiuses, and stopping before SEC/4. This action (procedure Central) transforms the configuration in one of class *Aperiodic*. In the third test, the algorithm checks if the configuration is *Equiangular*; if so, all robots move radially towards SEC eventually evolving into a *Regular* configuration. In the fourth test, if the configuration is *Pre-regular*, each robot moves towards its matching vertex in the supporting polygon. This action, called procedure Pre-regular is precisely the technique described in [8] to move from a



(a) Pre-equiangular (b) Landmark-co-radial (c) Antipodal-referees

(d) Post-periodic (e) Post-periodic (f) Post-periodic

Fig. 2. Examples of the possible evolutions of a *Periodic* configuration

Biangular configuration into a *Regular* one; during the action the configuration remains *Pre-regular* and it eventually evolves into *Regular*.

3.4 The Intermediate Tests

Having failed the initial tests, the next sequence of tests is for the classes of configurations defined below, which can occur as the initial configuration, or as an evolution from a *Periodic* configuration. Along with the definitions, the actions to perform in each configuration are given.

Pre-equiangular. There are robots both on SEC and on SEC/4, and nowhere else. The robots on SEC are at least three, and those on SEC/4 are forming an "almost" *Regular* configuration; that is, a *Regular* with one missing point for each robot on SEC. The missing points may be arranged in two different ways. They may form a "regular pairs" arrangement, in which there are pairs of missing points in adjacent positions, in such a way that the pairs are equally spaced around SEC/4; otherwise, they form a "regular pairs" arrangement in which exactly one element of each pair has been removed. There is a bijection between robots on SEC and missing points, determined by the minimum total distance the robots on SEC must travel to occupy them (Figure 2(a) shows an arrangement on SEC/4 of the second type).

In this case, the robots on SEC rotate towards their targets, which are uniquely determined by the positions of the robots on SEC/4. With this action, called procedure PRE-EQUIANGULAR, the robots eventually reach an *Equiangular* configuration.

Landmark-co-radial. The robots on SEC form an *Equiangular* set, and these are the referees, which also coincide with the landmarks. The landmarks define the n target half-lines, in such a way that either all landmarks lie on some targets (as in Figure 2(b)), or they lie on bisectors of adjacent targets. All the non-referee robots are on or inside SEC/4: each robot on SEC/4 is on a target; the only ones *strictly* inside are those co-radial with the referees, and at most one robot (called *walker*) for each referee. The central targets of each sector defined by two adjacent referees are all occupied by robots on SEC/4 in such a way that, for each landmark, the open sector Γ defined by the nearest target in the clockwise direction that is occupied by a robot on SEC/4 and the nearest one in the counterclockwise direction contains as many robots as targets. Moreover, Γ contains at most one walker, and the targets in Γ that lie to the left of the landmark differ by at most one unit from those to the right. A *co-radial Biangular* configuration falls in this class, too.

In this configuration, the intended behavior is to "resolve" all the robots that are co-radial to the referees, and have them move to their targets, reaching an *Equiangular* or a *Pre-equiangular* configuration (depending whether the referees are already on their targets or not).

Note that in a *Landmark-co-radial* the only unoccupied targets correspond to the groups of co-radial robots of the landmarks and to at most one robot per landmark, the *walker*, which is moving towards a target. The co-radial robots move in turns. If there is no walker in the sector Γ (as defined above) around a landmark, the most internal non-referee that is co-radial with that landmark rotates toward the farthest away target among those in Γ, becoming a *walker*. When a walker reaches its target, it moves radially to reach SEC/4. If all the non-referees are on their targets, they all lie on SEC/4, and the configuration happens to be *Antipodal-referees* (see below), then the two non-referees closest to the landmarks move toward SEC (thus "forcing" the configuration to transition into an *Antipodal-referees* configuration that is not a *Landmark-co-radial* anymore, which is tested after *Landmark-co-radial* by the algorithm). Otherwise, the configuration becomes either *Equiangular* or *Pre-equiangular*, as intended.

Post-periodic. The robots on SEC form an *Equiangular* or a *Simple biangular* set, and they are the referees. All other robots lie on SEC/4 or inside of it. If the referees are *Equiangular*, the landmarks coincide with the referees and they all have the same number of co-radial robots, which lie *strictly* inside SEC/4. If the referees are *Biangular*, the landmarks are the midpoints of neighboring referees, and no robot is co-radial with any landmark. The robots that are not co-radial with the landmarks are equidistributed among the sectors defined by the landmarks.

In this configuration, the targets are calculated with respect to the landmarks, depending on the parity of the robots that are co-radial with each landmark (including the referees): if they are odd, then the landmarks lie on some targets (Figure 2(d)); if they are even (which includes zero), the landmarks lie on bisectors of adjacent targets (Figures 2(e) and 2(f)). Note that, if such co-radials are odd, the referees must be *Equiangular*. Each robot may be associated with a unique target, or to two possible targets (in case of left-right symmetry of its view).

The intended behavior in a *Post-periodic* configuration is to have all robots move onto SEC/4 on their respective targets, except for the robots that are co-radial with some landmark, thus reaching a *Landmark-co-radial* configuration. To do so, the non-referees that are not co-radial with the landmarks and that can reach SEC/4 without colliding with other robots, move radially toward it. If none can do it and there are co-radial robots that are not co-radial with any landmark, the most internal of these co-radials rotates in an arbitrary direction of 1/4 of the minimum non-zero element in μ. If all the non-referees that are not co-radial with the landmarks are already on SEC/4, they orderly rotate on SEC/4 until they reach their targets (which are now uniquely determined). This is done in such a way that only the robots that can reach their target without colliding with other robots move. Each move is cautious, with critical points corresponding to *Landmark-co-radial* and *Pre-equiangular* configurations. At this point, the configuration becomes: *Landmark-co-radial* if the referees are *Equiangular* and have co-radial robots; *Pre-equiangular* if the referees are *Biangular* and they are not on their targets; or *Equiangular* if the referees are *Equiangular* and there are not co-radial robots, or if they are *Biangular* and already on their targets.

<u>*Antipodal-referees*</u>. There are two antipodal robots on SEC, which are the referees. On SEC/4 there are (possibly among others) $n-4$ robots that are forming a *Regular* configuration with some missing points. More precisely there are two antipodal pairs of adjacent missing points, such that each referee is equidistant to two adjacent missing points. Furthermore, there are two other robots co-radial with two non-adjacent missing points, which lie between SEC/4 and SEC (possibly on SEC/4 or on SEC). Note that this configuration is uniquely identifiable and has period either n or $n/2$ (see Figure 2(c)). In this configuration, the robots closest to the referees (one for each referee) move towards SEC, eventually reaching a *Pre-equiangular* configuration.

<u>*Simple biangular*</u>. In this case, the intended behavior of the robots is to reach a *Pre-regular* configuration by moving toward SEC according to the cautious move protocol, with critical points on SEC/4 (where a *Landmark-co-radial* or a *Pre-equiangular* may be formed), and additional critical points where *Pre-regular* configurations may be formed (see Theorem 2). If the robots already on SEC belong to the same analogy class, the other robots in the same class move first.

3.5 The Periodic Test

<u>*Periodic*</u>. If the procedure PERIODIC is executed, it means that the configuration is *Periodic*, and additionally it does not belong to any of the classes described

above. In this case, the intended behavior is to elect the referees, define the landmarks, have the referees move onto SEC and the non-referees move into SEC/4, reaching a *Post-periodic* configuration. In trying to do this, the robots can find themselves in a variety of different configurations, and the algorithm might switch to several different cases.

Let k be the period. If there exist robots with exactly n/k homologous robots, then the lex-first among these robots are chosen to be the referees, as well as the landmarks. If this is not the case, all homology classes must have size exactly $2n/k$. If the robots in $\mathcal{L}_1$ are not *Equiangular* (and therefore they are strictly *Biangular*), they are chosen to be the referees; otherwise the referees are the robots in $\mathcal{L}_2$. Note that in both cases the referees form a *Simple biangular* set; the landmarks are selected to be the midpoints of neighboring referees. Hence, by construction, the landmarks are always n/k points forming an *Equiangular* set (with respect to the center of the SEC of all robots), and they define n/k sectors, each containing the same number of robots in its interior.

If the configuration is *Double-biangular*, no referee is on SEC, and some non-referees are not on SEC, then all the non-referees move radially to reach SEC. Otherwise, if there are referees not on SEC, they move radially to reach SEC. If all the referees are on SEC, the other robots move radially inward until they reach SEC/4 or its interior. All non-referee robots that are co-radial with some landmark move *strictly* inside SEC/4. The non-referee robots move in turns, in such a way that only homologous robots can move together. Specifically, the non-referees that belong to homology classes of size n/k move first.

In all cases, all movements are cautious, with critical points on SEC/4 (which may yield a transition into *Landmark-co-radial*, *Antipodal-referees*, or *Pre-equiangular*), and those determined by *Pre-regular* configurations.

When this is done, the configuration becomes *Post-periodic*, with some exceptions: if the robots not co-radial with the referees are already on their targets on SEC/4, except at most one per landmark, the configuration becomes *Landmark-co-radial*; if the only robots not on their targets are the referees, and the referees are more than two, the configuration becomes *Pre-equiangular*; if the only robots not on their targets are the referees, and the referees are only two, the configuration becomes *Antipodal-referees*.

3.6 The Aperiodic Tests

In this last set of tests, *Post-aperiodic* and *Aperiodic* configurations are addressed. Similarly to the previous cases, the intended behavior of the actions is to elect the referees and to identify landmarks and targets. From the *Aperiodic* configuration, the intended behavior is to reach a *Post-aperiodic* configuration and, from there, an *Equiangular* configuration.

Post-aperiodic. There are either one or two robots on SEC/3, which are the referees. All other robots are found between SEC/2 and SEC. If there are two referees, they are not antipodal (i.e., their midpoint is not the center of SEC).

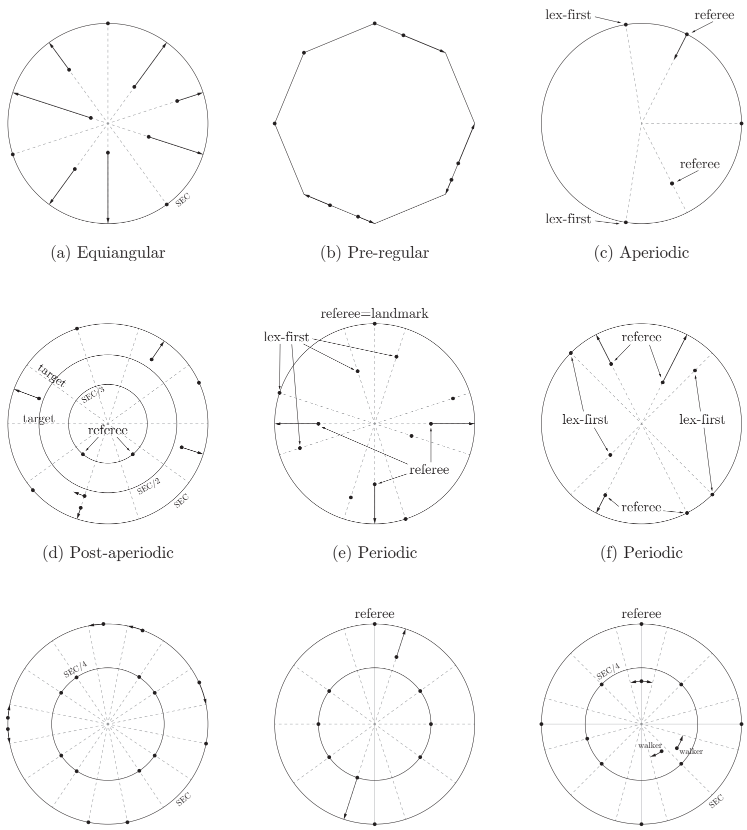
In this configuration, the actions taken by the robots (procedure Post-Aperiodic) are as follows. If there are two referees, and all the non-referees are on SEC forming a *Regular* set with two adjacent missing points, the two referees rotate on SEC/3 until they become co-radial with the missing points, and the configuration becomes *Equiangular*. Otherwise, the targets are identified by the referees on SEC/3, and a unique target is assigned to each robot. The non-referees that can move radially to SEC without colliding, do so. If there are non-referees that cannot radially move to SEC (because other robots are in their way), then the most internal non-referees rotate of 1/4 of the minimum non-zero element of μ to remove the co-radiality. If all the non-referees are on SEC and there is only one referee, the non-referees cautiously rotate to their respective targets, in such a way that SEC never changes and no two robots collide, and using *Simple biangular* and *Periodic* configurations as critical points. If the targets are reached, the configuration becomes *Equiangular*. Finally, if there are two referees, and all non-referees are on SEC, not forming a *Regular* set with two adjacent missing points, the non-referees rotate on SEC with a cautious move as in the previous case, with additional critical points given by the configurations in which the robots on SEC form a *Regular* set with two adjacent missing points. In this last case, the configuration may become *Simple biangular*, *Periodic*, *Equiangular*, or remain *Post-aperiodic*.

Aperiodic. The procedure APERIODIC is executed if the current configuration fails all previous tests. If the configuration is *co-radial uni-aperiodic*, then the lex-first is unique, and must have co-radial robots. In this case the referee is the most internal among the robots that are co-radial with the lex-first. If the configuration is *non-co-radial uni-aperiodic*, the lex-first is still unique, but it may be necessary to keep SEC intact. If this is not the case, the lex-first is the referee, otherwise the referee is the lex-second (it is easy to see that, if $n \geqslant 5$, one of these two robots can be removed without altering SEC).

If the configuration is *co-radial bi-aperiodic*, let r and r' be, among the robots that are co-radial with the lex-first robots, the most internal ones, respectively. If r and r' are not aligned with the center of SEC, then they are chosen to be the referees. Otherwise, the referees are the first two robots in the lexicographically minimum order (which are homologous) that can be *safely* removed without altering SEC (assuming that all robots that can reach SEC radially are already on SEC), and such that they are the most internal robots among their co-radials. (Note that, in some configurations, these referees happen to be the same robot. In these cases, the referee is unique.)

Finally, if the configuration is *non-co-radial bi-aperiodic*, the referees are the first two (just one, in some special cases) homologous robots that are not aligned with the center of SEC, and such that, when all robots are on SEC, they can be removed without changing SEC.

The non-referees that are inside or on SEC/2 move out of SEC/2. Those that can reach SEC without colliding, do so. They take turns in such a way that only homologous robots can move together (hence at most two), and they move radially outward, performing a cautious move with critical points on SEC/4, SEC/3, SEC/2, SEC, and those determined by the *Pre-regular* configurations (see Theorem 3). During these movements, the configuration may become ei-



(a) Equiangular (b) Pre-regular (c) Aperiodic
(d) Post-aperiodic (e) Periodic (f) Periodic
(g) Pre-equiangular (h) Antipodal-referees (i) Landmark-co-radial

Fig. 3. Some examples of configurations

ther *Post-periodic*, *Landmark-co-radial*, *Antipodal-referees*, or *Pre-equiangular* (when they pass through SEC/4 or when they reach SEC), or *Post-aperiodic* (when they pass through SEC/3), or *Pre-regular*. Otherwise, the configuration stays *Aperiodic*. If all the non-referees are outside SEC/2 and none of them can move to SEC without colliding, the referees move and reach SEC/3. They use a cautious move with SEC/4 as a critical point, and those determined by the *Pre-regular* configurations (see Theorem 3). The configuration may become *Post-periodic*, *Landmark-co-radial*, *Antipodal-referees*, or *Pre-equiangular* (when the robots reach SEC/4), or *Pre-regular*. Otherwise it becomes *Post-aperiodic*, as intended.

4 Correctness

The correctness proof is quite lengthy and can be found in [14]. We give here only an intuition of the main ingredients.

To prove correctness, we need to analyze all possible transitions between configurations. Some transitions come as a result of the "intended" behavior of the robots executing the algorithm; other transitions come as "accidental" byproducts of the execution. The proof is then a detailed examination of all the possible executions of the algorithm in the space of robots' configurations, paying special attention to the transitions that may arise as critical points of cautious moves. In the following, let $\mathcal{R} = \{r_1, \cdots, r_n\}$ denote a swarm of $n > 4$ robots. let $r_i(t)$ denote the location of robot r_i at time $t \geqslant 0$, and $\mathcal{R}(t) = \{r_1(t), \cdots, r_n(t)\}$.

We first prove that robots executing the cautious move protocol indeed behaves as intended.

Theorem 1. *Let a swarm of n robots execute a cautious move with critical point set $\bigcup_{i=1}^{k} C_i$, with $|C_i| = n$ for $1 \leqslant i \leqslant k$, from an initial configuration in which no robot is moving. Then, during the cautious move, whenever the robots are found in a configuration C_i, they all stop in that configuration.*

We then analyze the behaviour of the algorithms with respect to the critical points; in particular, to the *Pre-regular* case, which turns out to be the hardest to treat.

Lemma 1. *If S is* Pre-regular*, then it is not* Central*,* Post-periodic*,* Landmark-co-radial*,* Antipodal-referees*,* Pre-equiangular*, nor* Post-aperiodic*.*

Theorem 2. *Let $\mathcal{R}(0)$ be a* Simple biangular *configuration, $n > 4$, and let the robots execute procedure* Simple Biangular *with suitable critical points. Then, the robots eventually reach a* Pre-regular *configuration, and they stop as soon as they reach it.*

Theorem 3. *Let $\mathcal{R}(0)$ be an* Aperiodic *configuration, $n > 4$, and let the robots execute procedure* Aperiodic*. Then, as soon as they reach a* Pre-regular *or a* Simple biangular *or a* Aperiodic *configuration, they all stop in that configuration.*

The previous set of Theorems guarantees the correct execution of cautious moves. We conclude showing that the directed graph of configurations and their transitions (depicted in Figure 4) contains no cycles, and the only sink is the *Regular* configuration.

Lemma 2. *If $n > 4$, no transition is possible other than those illustrated in Figure 4.*

Theorem 4. *The* Uniform Circle Formation *problem is solvable by $n \neq 4$ robots in $\mathcal{ASYNC}$.*

The case $n = 4$ is still open.

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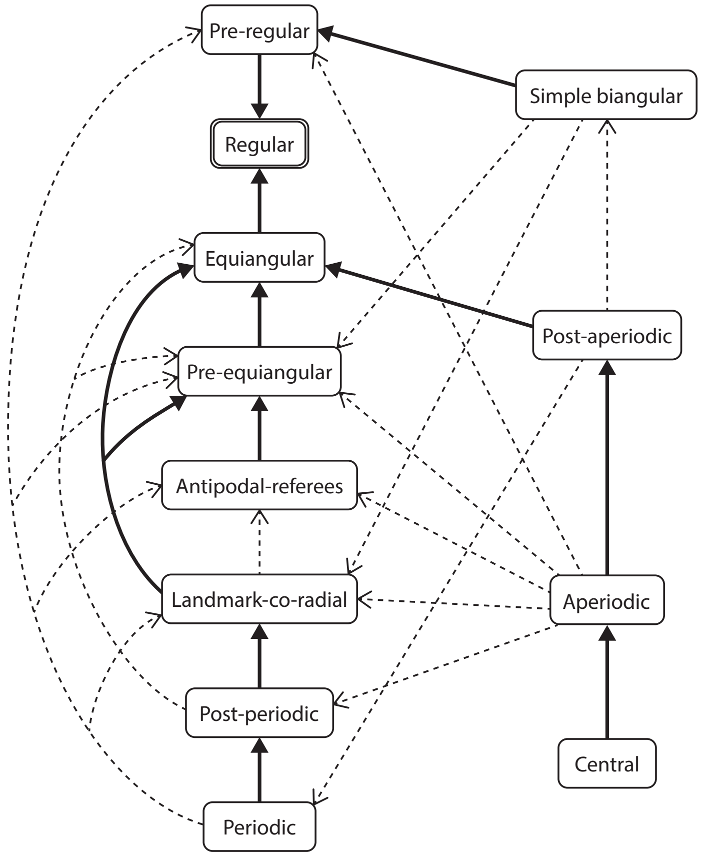

Fig. 4. Configurations, with their intended transitions (thick arrows) and incidental transitions (dashed arrows)

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