SENSE OF DIRECTION, TOPOLOGICAL AWARENESS AND COMMUNICATION COMPLEXITY

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ABSTRACT

Based on some recent results, it is here argued that the communication complexity of distributed problems can be greatly affected by two factors hereby identified as 'sense of direction' and 'topological awareness'. It is also suggested that 'insensitivity' to either or both factors is an indicator of the inherent difficulty of a distributed problem. A bibliography of recent results is included.

1. INTRODUCTION

Recently, distributed algorithms have been developed for solving a variety of problems, ranging from determination of graph properties to the distributed dictionary problem. The underlying model in most of these investigations is that of a point-to-point or neighbourly network [35,36]: the network is described as an undirected graph G where nodes represent processors, and arcs represent direct communication links between processors: each processor is aware of and can transmit messages only to the processors to which it is directly connected (the neighbours); the transmission of a message on an arc constitutes a communication activity. A computation in this model can be started at one or several nodes (the sources) and the result of the computation (more precisely, the projection of the result [36]) will be known at one or more nodes (the sinks); it is assumed that the behaviour of each processor during the computation is symmetric (i.e., every processor executes the same identical algorithm). A computation will in general require both processing (performed locally) and communication activities; processing time is assumed to be negligible when compared to transmission and queueing delays. For a more detailed characterization of the model, see any of the references in the enclosed bibliography.

Within this model, lower and upper bounds have been established on the amount of communication activities required to solve certain problems (the communication complexity). These bounds differ depending on the topology of the network (i.e., the graph G).

It has been long 'suspected' that the communication complexity also depends on the kind and amount of 'topological information' available to the nodes.

In this note, I argue that this suspicion has been confirmed by some recent results. Namely, these results show that two 'topological information' factors can directly and greatly affect the communication complexity. These key factors are hereby identified as sense of direction and topological awareness.

2. SENSE OF DIRECTION

2.1. BIRTH OF A SUSPICION: ELECTION IN A RING

Among distributed problems, one of the most investigated is the election (or extrema-finding) in a ring of processors:

> the graph G is a ring of n nodes; each node has a distinct value of which it alone is aware; we want to design an algorithm (to be made available to all nodes) which, once executed, will determine the node having the largest value.

Any node can be a source (i.e., all nodes may simultaneously start the execution of the algorithm); the node with highest value is the sole sink. Constraints exist on what can be included in a message, typically, an identifier and $O(\log n)$ bits. [1,4,6,10,11,16,20,24].

The first $O(n \log n)$ solution is due to Hirshberg and Sinclair [16]. Their solution allows messages to be sent in both directions along the ring, and does not rely on any global sense of direction; i.e., "left" may not have the same meaning to all processors. Burns [4] proved that $\Omega(n \log n)$ communication activities are needed if no global sense of direction is assumed.

Other investigators, motivated by a conjecture by Hirshberg and Sinclair, presented $O(n \log n)$ election algorithms which use only one direction in the ring whose orientation is globally known (the unidirectional case); i.e., "left" has the same meaning to all nodes and messages can only be sent to the "left" [10,29]. Pachl, Korach and Rotem [27] proved that

$\Omega(n \log n)$ communication activities are necessary for the unidirectional case. Subsequent work concentrated on reducing the multiplicative constant in both cases [11,20,24].

A close look at the "best" bounds known to date for both cases (shown in Figure 1) shows that the bounds for the unidirectional case are "better" (i.e., upper-bound is smaller and lower bound is greater) than the ones for the bidirectional case with no global sense of direction. A possible explanation is that this "gap" is purely accidental (e.g., the analysis of the bidirectional version has not been as tight as it should, etc.). Another possible explanation is that the unidirectional version is "different" from (in the sense of "easier" than) the other version of the problem; assuming this to be the case, since the unidirectional version uses less communication power (recall, messages are always sent in only one direction), the source of the difference must lie in the availability of a global sense of direction.

Which explanation is correct is still not known. However, the latter approach leads to the following observation. Let q and $\bar{q}$ denote the presence and absence, respectively, of global sense of orientation; b and u denote bidirectional and unidirectional communication capabilities, respectively; and $\varepsilon_R(x,y)$ denote the communication complexity of the election problem in a circle when conditions $x \in \{q,\bar{q}\}$ and $y \in \{b,u\}$ occur. Then the following relations trivially hold:

$$\varepsilon_R(q,u) \geq \varepsilon_R(q,b)$$

$$\varepsilon_R(\bar{q},b) \geq \varepsilon_R(q,b)$$

That is, the election problem with both bidirectional communication capabilities and global sense of direction is <u>easier</u> than the other two versions. In particular, the bidirectional election problem is <u>easier</u> if a global sense of direction exists. How much easier is not known; I suspect that $\varepsilon_R(q,b) = \Omega(n \log n)$, so that the only relevant difference will occur in the multiplicative constant.

Since in a circle both minimum-weight spanning-tree and simple spanning-tree construction are 'equivalent' to the election problem [36], the above results and observations apply also to these two problems.

This is the first indication that (the presence of a) sense of direction influences (positively) the communication complexity of a problem.

2.2. HARD EVIDENCE: ELECTION IN A COMPLETE GRAPH

From the previous discussion, it follows that for any given graph topology T

$$\varepsilon_T(\bar{q},b) \geq \varepsilon_T(q,b)$$

My aim is to show that, for some T, $\varepsilon_T(\bar{q},b) >> \varepsilon_T(q,b)$.

Consider the case of G being a complete graph on n nodes; each node is aware of being in a complete graph, and can distinguish between its $n-1$ neighbours; i.e., each node has available a local labelling of edges. If no other assumption is made on the labelling, no global sense of direction is available to the nodes. Korach, Moran and Zack [19] have shown that in this case $\Omega(n \log n)$ message transfers are necessary to perform election; that is,

$$\varepsilon_C(\bar{q},b) = \Omega(n \log n).$$

Presence of a global sense of direction in a complete graph means that the local labelling are all 'globally consistent' (analogously, in a ring, it meant that "right" had the same meaning to all processors). A globally consistent labelling is, for example, the following:

i) a Hamiltonian cycle is identified; and

ii) a global sense of orientation on the cycle exists; and

iii) all arcs incident on a node are labelled according to the 'distance' between that node and the other incident node in the cycle.

This labelling is shown in Figure 2. Assuming the existence of this labelling, Matsushita [24] has shown that $O(n)$ message transfers suffice for the election problem; an entire collection of $O(n)$ election have been subsequently devised by Sack, Santoro and Urrutia [33]. Since n is an obvious lower-bound, it follows that

$$\varepsilon_C(q,b) = \Theta(n)$$

That is, <u>(presence of) a global sense of direction can greatly affect (in a positive sense) the communication complexity.</u>

3. TOPOLOGICAL AWARENESS

In the previous discussion, it has been assumed that the nodes knew the topological 'structure' of the graph; e.g., the graph is a ring. The graph is complete, etc. Note that knowledge of the structure does not imply knowledge of the topology (e.g., adjacency matrix); for obvious reasons, this kind of knowledge has been termed 'myopic' [36].

We shall refer to the availability to all nodes of this 'myopic' knowledge as <u>topological awareness</u>. In this section I want to show that topological awareness (or the lack of it) can play a determining role in the complexity of a problem.

We have seen before that, in a complete graph with topological awareness, election requires $\Omega(n \log n)$ or $\Omega(n)$ message exchanges depending on whether a global sense of direction is present. I have shown in [36] that, if no topological awareness exists, then the worst-case communication complexity $\varepsilon_{n,e}$ of the election problem ε over all graphs with n nodes and $e \geq n$ edges is bounded below by $\Omega(e + n \log n)$. Combined with the $O(e + n \log n)$ upper-bound by Gallager [14], this yields

$$\varepsilon_{n,e} = \Theta(e + n \log n)$$

for $e \geq n$. This, together with other observations stated in [36], leads to the conclusion that, <u>in a complete graph where the nodes are not aware of being in such a graph</u>, $O(n^2)$ message transfers are needed. In other words, absence of topological awareness pushes the lower-bound from $\Omega(n \log n)$ to $\Omega(n^2)$ for a complete graph.

This kind of 'sensitivity' of the communication complexity to topological awareness is not restricted to the election problem. Since the election and the spanning-tree construction problem $\mathcal{X}$ are 'equivalent' in absence of topological awareness [36], it follows that

$$\mathcal{X}_{n,e} = \Theta(e + n \log n); \qquad (2)$$

that is, $\Omega(n^2)$ message transfers are needed to construct a spanning-tree in a complete graph <u>without</u> topological awareness. However, Korach, Moran and Zacks have proved that, <u>with</u> topological awareness, the bound becomes $\Omega(n \log n)$. That is, also in this case, topological awareness plays an important role in the complexity.

4. PROBLEM DIFFICULTY

In [36], it has been shown that the minimum-weight spanning-tree construction problem W is 'reducible' to election problem ε if no topological awareness exists; i.e.,

$$W_{n,e} \geq O(\varepsilon_{n,e}).$$

In view of the $O(e + n \log n)$ upper-bound on $W_{n,e}$ by Gallager, Humblet and Spira [15], this implies

$$W_{n,e} = \Theta(e + n \log n) \qquad (3)$$

in absence of topological awareness.

From bounds (1)-(3), it follows that, with respect to order of magnitude, <u>spanning-tree construction, minimum-weight spanning-tree construction</u> and <u>election</u> are <u>equally 'difficult' in absence of topological awareness</u>.

From the $\Theta(n \log n)$ bounds for election and spanning-tree construction in rings and complete graphs with topological awareness, it follows that the quality of being 'equally difficult' is mantained for ε and $\mathcal{X}$ <u>regardless</u> of topological awareness (at least in such graphs).

On the other hand, minimum-weight spanning-tree construction **W** has been shown to require $\Omega(n^2)$ messages in complete graphs <u>with</u> topological awareness [19]. This seems to imply that **W** is actually a more difficult problem than and $\mathcal{L}$. Similarly, it seems to imply that the 'sensitivity' (or lack of it) of a problem to topological awareness and/or sense of direction might be an indication of the inherent difficulty of that problem.

Finally, to date it is still not known whether **W** in complete graphs is 'insensitive' also to sense of direction; my conjecture is that it is.

5. CONCLUSIONS AND OPEN PROBLEMS

The previous discussion seems to imply the following:

i) There is evidence that both topological awareness and sense of direction directly influence the complexity of problems in point-to-point models.

ii) The 'size' of the graph (e.g., the number of edges) is not as important as topological awareness; e.g., spanning-tree construction in a ring and in a complete graph.

iii) The availability of additional communication links can be exploited to reduce complexity only if there is a sense of direction; e.g., election in a complete graph vs. election in a ring.

iv) 'Sensitivity' (or lack of) to topological awareness and/or sense of direction can be an indicator of the problem's inherent difficulty within the point-to-point model.

The following problems are still open:

a) Prove (or disprove) the $\Omega(n \log n)$ bound on election in bidirectional rings with global sense of direction;

b) Prove (or disprove) the $\Omega(n^2)$ bound on minimum-weight spanning-tree construction in complete graph with global sense of orientation;

c) Determine an $a\, n \log n + O(n)$ upper bound on election in bidirectional rings with global sense of direction, where $a < 1$, and the algorithm achieving the bound exploits both bidirectionality and sense of orientation.

d) Prove that topological awareness affects the complexity of other distributed problems;

e) Prove that sense of direction affects the complexity of other distributed problems.

ADDENDUM

J. Urrutia, S. Zacks and I have just proved that a 'weak' sense of direction is sufficient to achieve $\Theta(n)$ bounds for both election and spanning-tree construction in complete graphs.
We have also proved that even with a 'weak' sense of direction, minimum-weight spanning-tree construction in complete graphs is still $\Omega(n^2)$. What happens with a 'strong' sense of direction (problem b above) is still not known. (See figure 3)

	UPPER BOUND	LOWER BOUND
UNIDIRECTIONAL (SENSE OF DIRECTION)	1.43 n log n + O(n) [10]	.69 n log n + O(n) [27]
BIDIRECTIONAL (NO SENSE OF DIRECTION)	1.89 n log n + O(n) [24]	.25 n log n + O(n) [4]

Figure 1. Existing bounds for election in a ring.

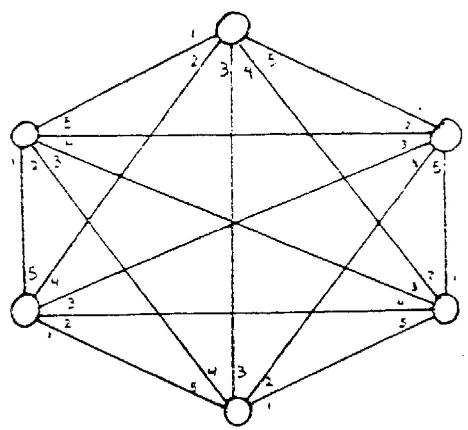

Figure 2. Global sense of direction in K_6

Election/Spanning-Tree	MST	
$\theta(n^2)$	$\theta(n^2)$	no topological awareness
$\theta(n \log n)$	$\theta(n^2)$	t.a., no sense of direction
$\theta(n)$	$\theta(n^2)$	t.a., 'weak' s.d.
$\theta(n)$	?	t.a., 'strong' s.d.

Fig. 3. Existing bounds for election, spanning-tree construction and minimum-weight spanning-tree construction (MST) in K_n

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